

1 Heisenberg Uncertainty Principle

In this section, we give a brief derivation and discussion of Heisenberg's uncertainty principle. The derivation follows closely that given in Von Neumann's *Mathematical Foundations of Quantum Mechanics* pp 230-232. The system we deal with is one dimensional with coordinate X ranging $-\infty, +\infty$. The uncertainty principle is a direct consequence of the commutation rule

$$[X, P] = i\hbar \quad (1)$$

While we need this operator equation to derive a concrete result, the general idea is present in any system where there are plane waves. A physically realizable wave is always in the form of a *wave packet* which is finite in extent. A wave packet is built up by superposing waves with definite wave number. By simple Fourier analysis, a highly localized packet will require a wide spread of wave vectors, whereas a packet with a large spacial extent can be composed of wave numbers quite close to a specific value.

We will carry out the derivation, following Von Neumann, using standard Hilbert space notation instead of Dirac notation. For expected values we write

$$\langle P \rangle \equiv \bar{P} = (\Psi, P\Psi), \quad \text{and} \quad \langle X \rangle \equiv \bar{X} = (\Psi, X\Psi).$$

Recall that the *length* of a vector in Hilbert space is defined by

$$\|\Psi\| \equiv \sqrt{(\Psi, \Psi)}$$

We will also need the *uncertainty* in X and P . For any observable O , the standard definition of uncertainty is

$$\Delta O \equiv \sqrt{(\Psi, (O^2 - \bar{O}^2)\Psi)},$$

which is a measure of the “spread” of values of O around its mean or expected value.

To make progress in the derivation, we need an expression which can be manipulated into one involving $[X, P]$. The expression used by Von Neumann is $(X\Psi, P\Psi)$. Using the fact that X and P are self-adjoint, can write

$$2i\text{Im}[(X\Psi, P\Psi)] = (X\Psi, P\Psi) - (P\Psi, X\Psi) \quad (2)$$

Now we using self-adjointness again, we can move all operators to the right slot, giving

$$(X\Psi, P\Psi) - (P\Psi, X\Psi) = (\Psi, XP\Psi) - (\Psi, PX\Psi).$$

Next we use the X, P commutator and we have

$$(X\Psi, P\Psi) - (P\Psi, X\Psi) = (\Psi, [X, P]\Psi) = i\hbar(\Psi, \Psi) = i\hbar, \quad (3)$$

where in the last equality we have assumed a normalizable state with $(\Psi, \Psi) = 1$. Comparing Eqs.(2) and (3) we now have

$$\text{Im}[(X\Psi, P\Psi)] = \frac{\hbar}{2} \quad (4)$$

So far we have made no approximations. Eq.(4) is exact. To get the uncertainty relation, we now make use of some inequalities. First we make use of the fact that the imaginary part of a complex number is less than the absolute value of the complex number. For us this says

$$\text{Im}[(X\Psi, P\Psi)] \leq |(X\Psi, P\Psi)|,$$

which implies using Eq.(4) that

$$|(X\Psi, P\Psi)| \geq \frac{\hbar}{2} \quad (5)$$

We now have an inequality involving X and P which came directly from the commutator Eq.(1), but it involves X and P in the same matrix element. To separate them, we now use the Schwarz inequality, which states

$$|(\Psi, \Phi)| \leq \|\Psi\| \cdot \|\Phi\|,$$

which is the Hilbert space version of the familiar statement that the dot product of two vectors is \leq the product of their lengths. For us, the vectors we will apply the Schwarz inequality to are $X\Psi$, and $P\Psi$. We have

$$|(X\Psi, P\Psi)| \leq \|X\Psi\| \cdot \|P\Psi\|,$$

so we now have

$$\|X\Psi\| \cdot \|P\Psi\| \geq \frac{\hbar}{2} \quad (6)$$

This is almost what we want, but makes no reference to the expected values \bar{X} and \bar{P} , both of which come into their respective uncertainties. To remedy this we define

$$X' = X - \bar{X}I, \quad P' = P - \bar{P}I.$$

These operators satisfy the basic commutation rule,

$$[X', P'] = i\hbar.$$

All the steps in our derivation hold if we replace X by X' , and P by P' , This allows us to write

$$\|X'\Psi\| \cdot \|P'\Psi\| \geq \frac{\hbar}{2}. \quad (7)$$

This really is the uncertainty principle. To see this, we write

$$\|X'\Psi\|^2 = (X'\Psi, X'\Psi) = (\Psi, X'X'\Psi) = (\Psi, (X^2 - 2X\bar{X} + \bar{X}^2)\Psi)$$

$$= (\Psi, (X^2 - \bar{X}^2)\Psi) = (\Delta X)^2,$$

and likewise for $\|P'\Psi\|^2$. Using these results we finally have

$$\Delta X \cdot \Delta P \geq \frac{\hbar}{2}$$

This completes the mathematical derivation of the uncertainty principle. There is a very nice physical discussion of it in Heisenberg's book *The Physical Principles of Quantum Theory* around page 20. As a historical note, Heisenberg only proved his principle for gaussian wave functions. The full Hilbert Space derivation given above was apparently first done by Hermann Weyl. Below is a photo of Heisenberg in his prime.

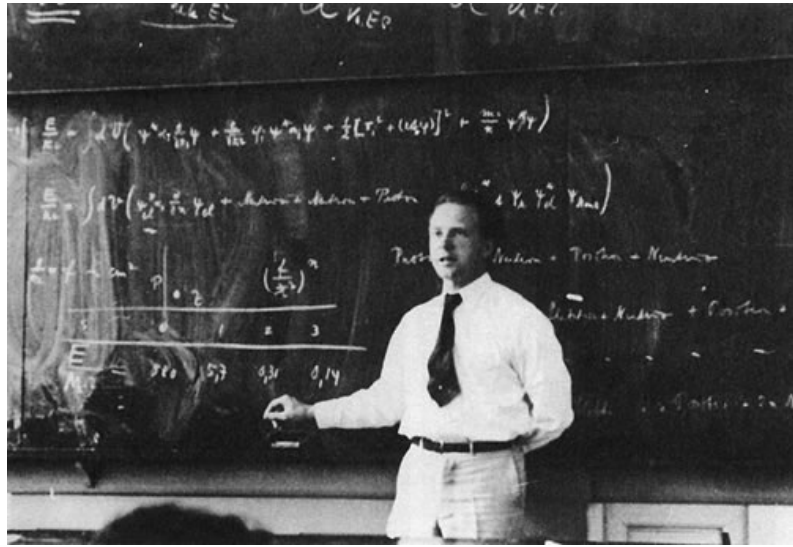


Figure 1: Young Heisenberg

Heisenberg's Measurement Uncertainty The above discussion does not say a word about measurements. It is instead a limit on the possible properties of a given state. Roughly, if $\langle x|\Psi \rangle$ has a narrow distribution, then $\langle p|\Psi \rangle$ will have a spread out distribution and vice versa. Heisenberg's original paper on uncertainty concerned a much more physical picture. The example he used was that of determining the location of an electron with an uncertainty δx , by having the electron interact with X-ray light. For an X-ray of wavelength λ , the best that can be done is

$$\delta x \sim \lambda.$$

But if an X-ray photon scatters from an electron, it will disturb the electron's momentum by an amount δp . We expect δp to be of order the X-ray photon's momentum, i.e.

$\delta p \sim h/\lambda$. The net result is that the product $(\delta x)(\delta p)$ should be of order Planck's constant, or

$$(\delta x)(\delta p) \sim h.$$

(We are not following factors of 2 or π at this point.)

A question that has been studied ever since Heisenberg's original paper is whether a precise inequality can be established for the case where actual measurements are made on a quantum system. As of October, 2013, this has not been definitively settled, but we can discuss the general framework presently being used to address the issue. We imagine that we have a quantum system \mathcal{S} . We take \mathcal{S} to have a coordinate X_s and conjugate momentum P_s , which satisfy

$$[X_s, P_p] = i\hbar.$$

The measurement of say X_s is done by coupling the quantum system \mathcal{S} to another system, the probe \mathcal{P} . In real experiments, \mathcal{P} will have many degrees of freedom. The final result of the interaction of \mathcal{P} with \mathcal{S} will be a record or reading. This final stage is classical, no issue of non-commuting operators or quantum fluctuations is involved at the end, i.e. the meter reads a certain number, period.

Theoretical analysis of measurement uncertainty involving a system \mathcal{S} and a probe \mathcal{P} have so far not included the case where \mathcal{P} has many degrees of freedom, the last stages of which are classical. Instead the probe is modeled as a second quantum system, whose quantum variables are somehow accessible to observation, and which can give information about the quantum variables of \mathcal{S} . This can be illustrated by denoting the \mathcal{S} quantum variables by X_s and P_s , and the \mathcal{P} variables by X_p, P_p .

The interaction of system and probe is described by an interaction or coupling Hamiltonian, H_c , which acts over a short time interval, T . Since the evolution of the total system is unitary, all operators at time T can be expressed in terms of the operators at time 0, the time at which H_c is turned on. The interaction Hamiltonian, H_c is chosen so $\langle X_p(T) \rangle = \langle X_s(0) \rangle$, i.e. by making a measurement of X_p at time T , we can deduce information about the operator X_s at time 0, i.e just before the coupling is turned on. There is inevitably some noise or measurement error introduced, and the so-called noise operator is defined for this case as

$$N = X_p(T) - X_s(0).$$

As a measure of the noise, the uncertainty or standard deviation of N is used,

$$\Delta(N) = \sqrt{\langle (X_p(T) - X_s(0))^2 \rangle}$$

In general, the momentum P_s of the system will be modified or disturbed by the interaction of \mathcal{S} with \mathcal{P} . The disturbance operator is defined by

$$D = P_s(T) - P_s(0).$$

The measure of disturbance is then

$$\Delta(D) = \sqrt{\langle (P_s(T) - P_s(0))^2 \rangle}.$$

The question then becomes, is

$$(\Delta(N))((\Delta D)) \geq \frac{\hbar}{2}?$$

No definitive answer is available at present. The discussion can be traced by looking at the article “Proof mooted for Quantum Uncertainty,” in **Nature** vol. 498, p 419, June 27, 2013.

Even if this controversy is finally resolved, it is clear that the above description of the probe is too simple. The case of a probe with many degrees of freedom, the final stages of which are classical or macroscopic, remains to be discussed in a satisfying way. Heisenberg’s uncertainty is likely to remain an alive subject for some time.