DIFFERENTIATION

🚩 SUPPOSE $f:(a,b)\to\mathbb{R}$, $c\in(a,b)$

SAY f IS DIFFERENTIABLE AT c AND HAS DERIVATIVE L AT c IF

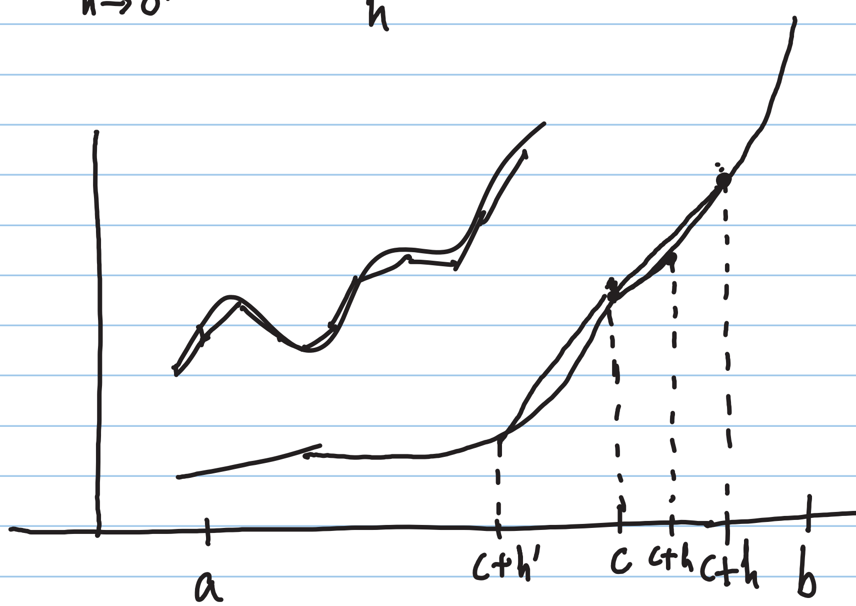
$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}=L.$$

🚩 LEFT DERIVATIVE OF f AT c:

$$\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h}$$

RIGHT DERIVATIVE OF f AT c:

$$\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$$



IF f IS DIFFERENTIABLE AT c, THEN f IS CONTINUOUS AT c.

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = L$$ GIVEN $\varepsilon > 0$, $\exists\, \delta > 0$ s.t

$$h \in (-\delta, \delta),\ h \neq 0 \quad L - \varepsilon < \frac{f(c+h) - f(c)}{h} < L + \varepsilon$$

$$h > 0 \Rightarrow (L-\varepsilon)h + f(c) < f(c+h) < f(c) + h(L+\varepsilon)$$

TAKE $\lim_{h\to 0}$: $(L-\varepsilon)h + f(c) \to f(c)$

$(L+\varepsilon)h + f(c) \to f(c)$

$$\Rightarrow \lim_{h\to 0^+} f(c+h) = f(c)$$. SIMILARLY $\lim_{h\to 0^-} f(c+h) = f(c)$

CONVERSE? NOT TRUE:

$f(x) = |x|$, $c = 0$.

$$\lim_{h\to 0^+} \frac{f(0+h)}{h} = \lim_{h\to 0^+} \frac{f(h)}{h} = 1.$$

BUT $$\lim_{h\to 0^-} \frac{f(0+h)}{h} = -1$$

SUPPOSE $f, g : (a,b) \to \mathbb{R}$, $a < c < b$, AND f, g ARE DIFFERENTIABLE AT c, THEN

(i) $(f \pm g)'(c) = f'(c) \pm g'(c)$

(ii) $(\alpha f)'(c) = \alpha f'(c)$

(iii) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$

(PRODUCT RULE)

(iv) $\left(\frac{f}{g}\right)'(c) = \dfrac{f'(c)g(c) - f(c)g'(c)}{g^2(c)}$

(QUOTIENT RULE)

(iii) $$\lim_{h\to 0} \frac{(fg)(c+h) - (fg)(c)}{h} =$$

$$\frac{f(c+h)g(c+h) + \left(f(c)g(c+h) - f(c)g(c+h)\right) - f(c)g(c)}{h}$$

$$= g(c+h)\left\{\frac{f(c+h) - f(c)}{h}\right\} + f(c)\left\{\frac{g(c+h) - g(c)}{h}\right\}$$

(1) (2)

$$\boxed{g(c) \cdot f'(c) + f(c) \cdot g'(c)}$$

EXAMPLES

(i) $f(x) = x^n, \; n \in \mathbb{N}$.

$$\lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0} \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-1}xh^{n-1} + h^n}{h}$$

$$= \lim_{h\to 0} nx^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + h^{n-1}$$

$$= nx^{n-1}$$

(ii) $f(x) = \sin x \quad f'(x) = \cos x$:

$$\lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{2\cos(x+h/2)\sin\left(\frac{h}{2}\right)}{h}$$

$$= \lim_{h\to 0} \cos\left(x+\frac{h}{2}\right) \cdot \lim_{h\to 0} \frac{\sin(h/2)}{(h/2)}$$

$$= \cos x \cdot 1$$ SINCE

(i) $\cos x$ is CONTINUOUS

(ii) $$\lim_{\eta\to 0} \frac{\sin\eta}{\eta} = 1$$

$$\Rightarrow (\sin x)' = \cos x.$$

ANOTHER WAY TO LOOK AT f'

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$$

i.e

IF h IS "SMALL ENOUGH"

$$\frac{f(x+h)-f(x)}{h} \approx f'(x) \quad \text{(APPROX. EQUAL)}$$

$$f(x+h)-f(x) \approx f'(x)h$$

$$\rightarrow \boxed{f(x+h) \approx f(x) + f'(x)h}$$

MORE FORMALLY,

$$\frac{f(x+h)-f(x)}{h} - f'(x) =: \varepsilon_x(h)$$

THEN $\lim_{h\to 0} \varepsilon_x(h) = 0$

SO, WE HAVE

$$f(x+h) = f(x) + hf'(x) + h\varepsilon(h)$$

$\forall \; h \in (-\delta, \delta)$ FOR A SMALL $\delta > 0$.

CHAIN RULE

SUPPOSE f IS DIFFERENTIABLE AT c,
AND g IS DIFFERENTIABLE AT $f(c)$,
THEN $g \circ f$ IS DIFFERENTIABLE AT c
AND

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$$

EXAMPLE:

$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

$$\Rightarrow (\cos x)' = (\sin)'\left(\frac{\pi}{2} - x\right) \cdot \left(\frac{\pi}{2} - x\right)'$$

$$= \cos\left(\frac{\pi}{2} - x\right) \cdot (-1)$$

$$= -\sin x$$

THIS (USING THE EARLIER RULES) ALLOWS US TO CALCULATE DERIVATIVES OF POLYNOMIALS, TRIGONOMETRIC FUNCTIONS, AND THEIR COMBINATIONS.