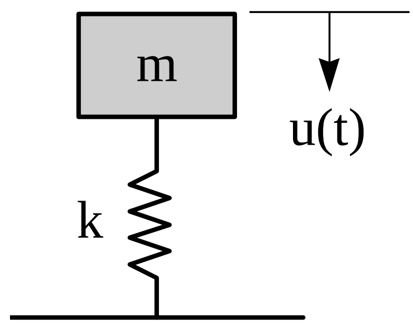
Free Vibrations

We begin by examining the response of the single-degree-of-freedom (SDOF) system with no external forces. The mass is set into motion by an initial displacement from its at rest position and/or an initial velocity. We will consider two systems, one with no means of dissipating energy and another with a viscous damping in the form of a dashpot.

Undamped Free Vibrations

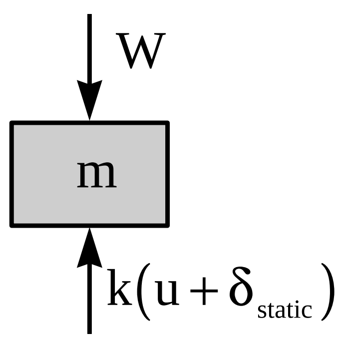
Consider the single-degree-of-freedom (SDOF) system shown at the right that has only a spring supporting the mass. If we examine a free-body diagram of the mass we see that the forces acting on it include gravity (the weight) and the resistance provided by the spring.



We use (1) Hooke's law (F = ku) and Newton's second law (F = ma) to write the following:

$$W - k\left(u + \delta_{static}\right) = m\ddot{u} \qquad (1)$$

where δ_{static} is the displacement of the mass when the system is at rest. The additional component of the displacement, u, is measured relative to this "at rest" position. The static displacement is related to the weight by:



$$W = k\delta_{static} \qquad (2)$$

Thus, Eq. 1 simplifies to:

$$m\ddot{u} + ku = 0 \qquad (3)$$

which is the *equation of motion* for the undamped SDOF system. The objective is to solve the equation of motion to determine the displacement of the mass as a function of time, u(t), subject to the initial conditions of the system.

One approach to solving this partial differential equation is to assume a solution of the form:

$$u(t) = Ae^{rt} \qquad (4)$$

Take the second derivative with respect to time of Eq. 4 yielding:

$$\ddot{u}(t) = r^2 A e^{rt} \tag{5}$$

and substitute Eqs. 4 and 5 into Equation 3 giving:

$$m r^2 A e^{rt} + k A e^{rt} = 0 \tag{6a}$$

which simplifies to:

$$m r^2 + k = 0 \tag{6b}$$

Solving for r produces two possible roots:

$$r = \pm i\sqrt{\frac{k}{m}} \tag{7}$$

where $i = \sqrt{-1}$. Thus the solution is given by:

$$u(t) = A e^{i\sqrt{\frac{k}{m}}t} + B e^{-i\sqrt{\frac{k}{m}}t} \tag{8}$$

We define the radical term in the exponent to be the *circular natural frequency* of vibration of the system:

$$\omega_n = \sqrt{\frac{k}{m}} \text{ in rad/sec} \tag{9}$$

Note that the circular natural frequency is related to the *natural frequency* by:

$$f_n = \frac{\omega_n}{2\pi} \tag{10}$$

and the *natural period* by:

$$T_n = \frac{1}{f_n} \tag{11}$$

To evaluate the coefficients A and B in Eq. 8, we need two initial conditions. Usually, these are the initial displacement and velocity of the mass:

$$u(t = 0) = u_0 \tag{12a}$$

$$\dot{u}(t=0)=\dot{u}_0 \tag{12b}$$

After substituting these expressions into Eq. 8 and its first derivative, we obtain:

$$u(t)=\left(\frac{u_0}{2}+\frac{i\dot{u}_0}{2\omega_n}\right)e^{-i\omega_n t}+\left(\frac{u_0}{2}-\frac{i\dot{u}_0}{2\omega_n}\right)e^{i\omega_n t} \tag{13}$$

Figure 1 shows the displacement time history of this SDOF with the initial conditions of $u_0 = 1.0$ and $\dot{u}_0 = 0.0$. Note that the amplitude of the displacement does not diminish with time because there is no means of attenuating energy within the system.

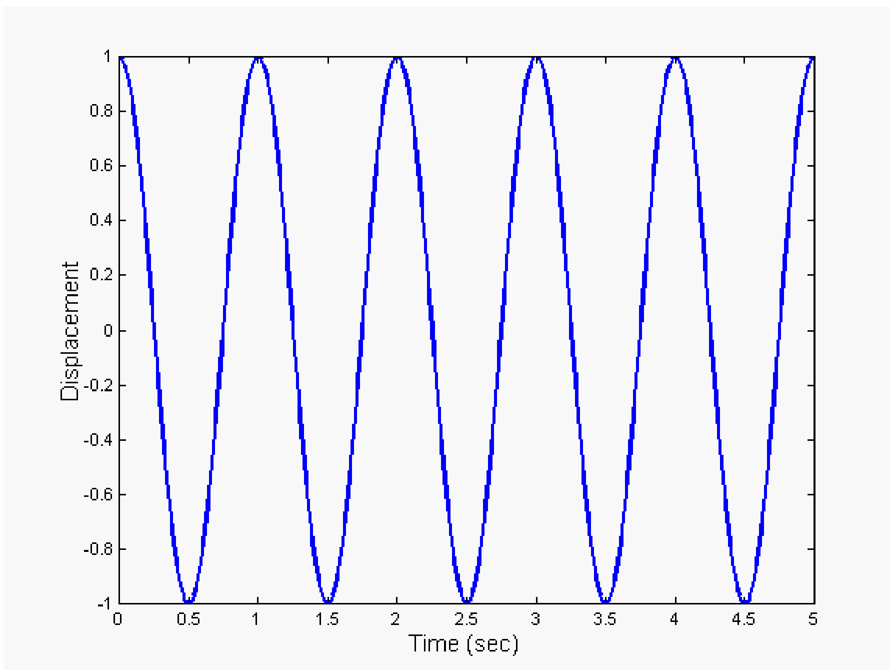
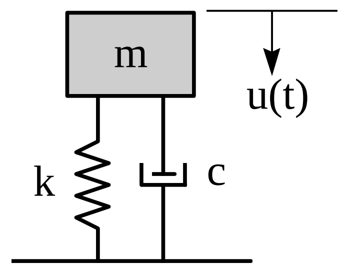


Figure 1 Displacement Time History for Free Vibrations of an Undamped SDOF System

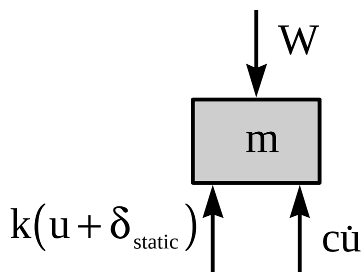
Damped Free Vibrations

Consider the single-degree-of-freedom (SDOF) system shown at the right that has both a spring and dashpot. If we examine a free-body diagram of the mass we see that an additional force is provided by the dashpot. The force is proportional to the velocity of the mass.



$$F_{damping} = c\dot{u} \tag{14}$$

where c is the viscous dashpot coefficient. Summing forces in the vertical direction yields:



$$W - k(u + \delta_{static}) - c\dot{u} = m\ddot{u} \tag{15}$$

After simplifying using Eq. 2, the equation of motion of the system is:

$$m\ddot{u} + c\dot{u} + ku = 0 \tag{16a}$$

or

$$\ddot{u} + \frac{c}{m}\dot{u} + \frac{k}{m}u = 0 \tag{16b}$$

To solve this differential equation, assume a solution of the form:

$$u(t) = Ae^{rt} \tag{17}$$

Take the first and second derivative with respect to time of Eq. 17 yielding:

$$\dot{u}(t) = rAe^{rt} \tag{18a}$$

$$\ddot{u}(t) = r^2Ae^{rt} \tag{18b}$$

and substitute Eqs. 18a and b into Eq. 16b giving:

$$r^2Ae^{rt} + \frac{c}{m}rAe^{rt} + \frac{k}{m}Ae^{rt} = 0 \tag{19a}$$

which simplifies to:

$$r^2 + \frac{c}{m} r + \frac{k}{m} = 0 \tag{19b}$$

Solving for r produces two possible roots:

$$r = \frac{-c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \tag{20}$$

Thus the solution is given by:

$$u(t) = Ae^{r_1 t} + Be^{r_2 t} \tag{21}$$

We define a *critical value of c* such that the term inside the radical equals 0:

$$c_{crit} = 2\sqrt{km} \tag{22}$$

and a *fraction of critical damping*:

$$\beta = \frac{c}{c_{crit}} = \frac{c}{2m\omega_n} \tag{23}$$

Rearranging Eq. 23 yields:

$$\frac{c}{2m} = \omega_n \beta \tag{24}$$

which can be substituted into Eq. 20 to yield:

$$r = -\omega_n \beta \pm i\omega_n \sqrt{1-\beta^2} \tag{25}$$

After substituting, Eq. 21 becomes:

$$u(t) = e^{-\omega_n \beta t}\left[Ae^{-i\omega_n \sqrt{1-\beta^2}\,t} + Be^{i\omega_n \sqrt{1-\beta^2}\,t}\right] \tag{26}$$

To evaluate the coefficients A and B in Eq. 26, we need two initial conditions. Usually, these are the initial displacement and velocity of the mass:

$$u(t = 0) = u_0 \tag{27a}$$

$$\dot{u}(t = 0) = \dot{u}_0 \tag{27b}$$

After substituting these expressions into Eq. 26 and its first derivative, we obtain:

$$u(t) = e^{-\omega_n \beta t}\left[\left(\frac{u_0}{2} + \frac{i\left(\dot{u}_0 - \omega_n \beta u_0\right)}{2\omega_n\sqrt{1-\beta^2}}\right)e^{-i\omega_n\sqrt{1-\beta^2}\,t} + \left(\frac{u_0}{2} - \frac{i\left(\dot{u}_0 - \omega_n \beta u_0\right)}{2\omega_n\sqrt{1-\beta^2}}\right)e^{i\omega_n\sqrt{1-\beta^2}\,t}\right] \tag{28}$$

Consider the solution for three different values of β:

1. **β = 1 (critically damped)**

For β = 1 Eq. 25 reduces to:

$$r = -\omega_n \tag{29}$$

and the partial differential equation has repeated roots. As a result, the solution takes the form:

$$u(t) = \left[u_0 + \left(\dot{u}_0 + \omega_n u_0\right)t\right]e^{-\omega_n t} \tag{30}$$

Figure 2 shows the response of a critically damped SDOF for three initial conditions. The initial displacement is equal to 1.0, but the initial velocity of the mass is 1.0, 0.0, and –1.0 for the three different cases. Notice that the motion quickly diminishes to zero because of the large damping in the system.

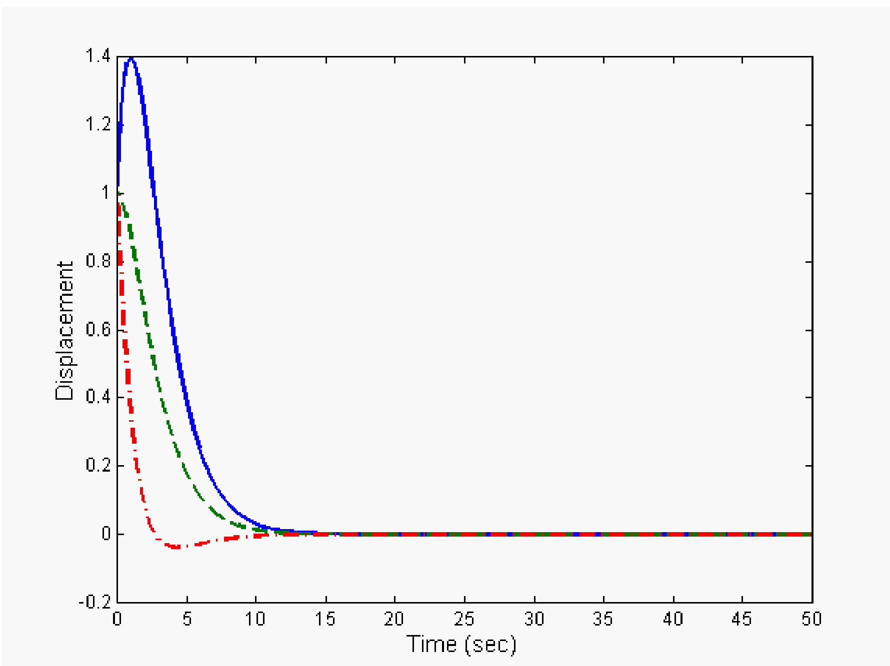


Figure 2 Displacement Time History of a Critically Damped SDOF System

2. **β > 1 (overdamped)**

The overdamped case is similar to the critically damped case. Eq. 28 can be used directly since the roots are not repeated. Figure 3 below shows the response of an overdamped SDOF for the same three initial conditions in Fig. 2. As in the case of the critically damped SDOF, the displacement quickly diminishes to zero because of the large damping in the system.

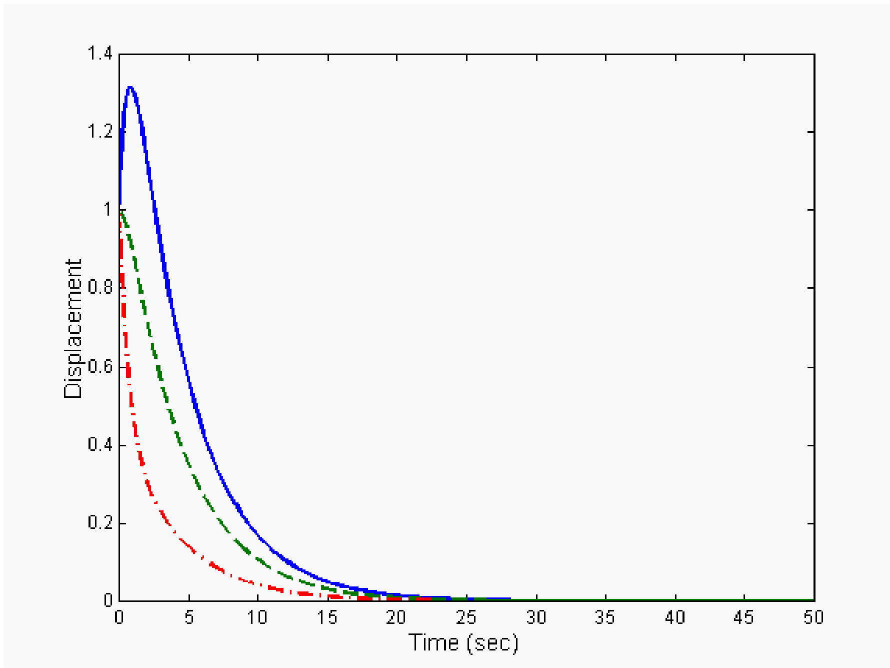


Figure 3 Displacement Time History of an Overdamped SDOF System

3. **β < 1 (underdamped)**

The case of most interest to us in soil dynamics problems is that in which the fraction of critical damping is less than 1.0. Equation 28 may be used again to calculate the displacement as a function of time. An example of a typical displacement time history for an underdamped SDOF is shown in Fig. 4.

Notice that the displacement time history continues for many cycles of motion (i.e. oscillatory). The motion can be considered to be a harmonic function modulated by a decreasing exponential function. The origin of these two functions can be seen in Eq. 28.

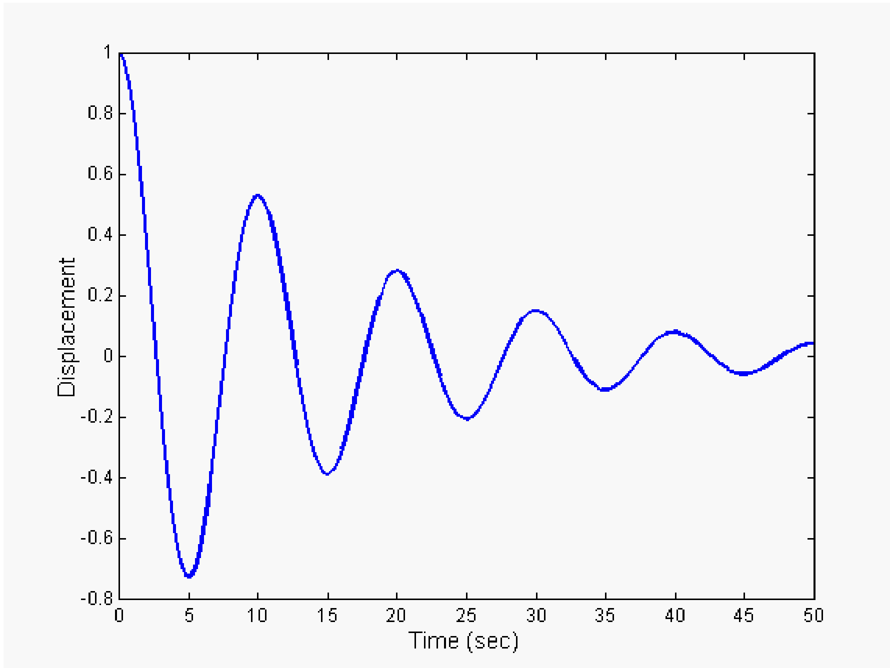


Figure 4 Displacement Time History of an Underdamped SDOF System

It is common to define the damped circular natural frequency as:

$$\omega_d = \omega_n \sqrt{1-\beta} \tag{31}$$

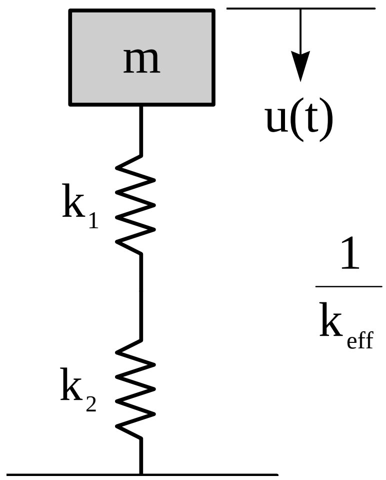
along the corresponding damped natural frequency and damped natural period, f_d and T_d, respectively. For small values of β, $\omega_d \approx \omega_n$.

Another widely used measure of the damping in a viscous system is the logarithmic decrement:

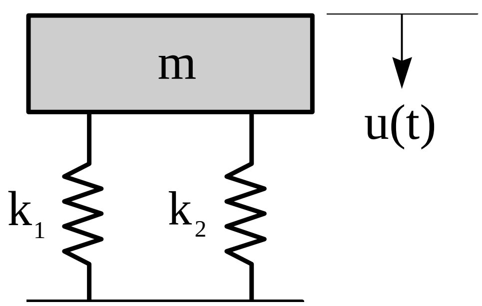
$$\delta = \frac{1}{n}\ln\left(\frac{u(t)}{u(t+nT_d)}\right) = \frac{2\pi\beta}{\sqrt{1-\beta^2}} \cong 2\pi\beta \text{ for small } \beta \tag{32}$$

Parallel and Series Springs

It will sometimes be necessary to consider springs acting in parallel or series. This is easily handled by determining the effective stiffness of a single, equivalent spring as shown below.



$$\frac{1}{k_{eff}} = \frac{1}{k_1} + \frac{1}{k_2}$$



$$k_{eff} = k_1 + k_2$$