MA108 ODE: Solving First order ODE's Lecture 2 (D2)

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What is a solution?

Consider $F(x, y, y', \ldots, y^{(n)}) = 0$. We assume that it is always possible to solve a differential equation for the highest derivative, obtaining

$$y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$$

and study equations of this form. This is to avoid the ambiguity which may arise because a single equation $F(x, y, y', \ldots, y^{(n)}) = 0$ may correspond to several equations of the form $y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$. For example, the equation $y'^2 + xy' + 4y = 0$ leads to the two equations

$$y' = \frac{-x + \sqrt{x^2 - 16y}}{2} \text{ or } y' = \frac{-x - \sqrt{x^2 - 16y}}{2}.$$

Definition

A explicit solution of the ODE $y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$ on the interval $\alpha < x < \beta$ is a function $\phi(x)$ such that $\phi', \phi'', \cdots, \phi^{(n)}$ exist and satisfy

$$\phi^{(n)}(x) = f(x, \phi, \phi', \cdots, \phi^{(n-1)}),$$

for every x in $\alpha < x < \beta$.

Implicit solution & Formal solution

Definition

A relation $g(x, y) = 0$ is called an implicit solution of $y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$ if this relation defines at least one function $\phi(x)$ on an interval $\alpha < x < \beta$, such that, this function is an explicit solution of $y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$ in this interval.

Examples:

1. $x^2 + y^2 - 25 = 0$ is an implicit solution of $x + yy' = 0$ in $-5 < x < 5$, because it defines two functions
$$\phi_1(x) = \sqrt{25 - x^2},\ \phi_2(x) = -\sqrt{25 - x^2}$$
which are solutions of the DE in the given interval. Verify!
2. Consider $x^2 + y^2 + 25 = 0$. We say $x^2 + y^2 + 25 = 0$ formally satisfies $x + yy' = 0$. But it is NOT an implicit solution of DE as this relation doesn't yield ϕ which is an explicit solution of the DE on any real interval I.

First order ODE & Initial Value Problem for first order ODE

We now consider first order ODE of the form $\boxed{F(x, y, y') = 0}$ or $\boxed{y' = f(x, y)}$.

Consider a linear first order ODE of the form $\boxed{y' + a(x)y = b(x)}$.

If $b(x) = 0$, then we say that the equation is homogeneous.

Definition

Initial value problem (IVP) : A DE along with an initial condition is an IVP.

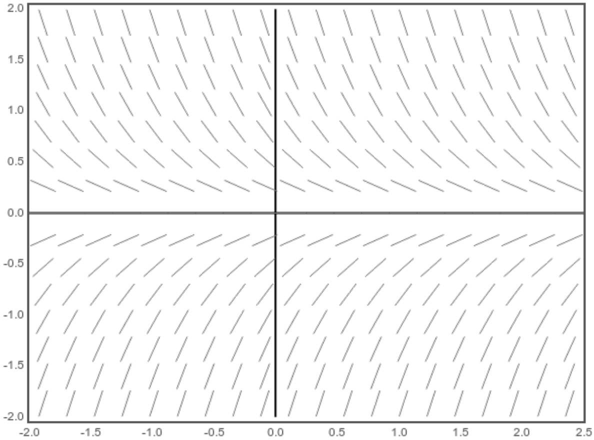
$$y' = f(x, y), \; y(x_0) = y_0.$$

Geometrical meaning : $\frac{dy}{dt} = -2 \cdot y$

1. Fix $k = 2$.
2. Suppose that y has certain value. From the RHS of the DE, we obtain $\frac{dy}{dt}$. For instance, if $y = 1.5$, $\frac{dy}{dt} = -3$. This means that the slope of a solution $y = y(t)$ has the value -3 at any point where $y = 1.5$.
3. Display this information graphically in ty-plane by drawing short line segments or arrows at several points on $y = 1.5$.
4. Similarly proceed for other values of y.
5. The figure given in the next slide is an example of a direction field or a slope field.

Slope field

Direction field for $\frac{dy}{dt} = -2y$.



Geometric Meaning of Solutions of $y^1 = f(x, y)$

Consider the first order ODE

$$\frac{dy}{dx} = f(x, y).$$

Suppose that $f(x, y)$ is defined in a region $D \subseteq \mathbb{R}^2$. If $y = \phi(x)$ is a solution curve and (x_0, y_0) is a point on it, then the slope at (x_0, y_0) is $f(x_0, y_0)$.
At each point $(a, b) \in D$, assign a vector with slope $f(a, b)$. The vector field $H : D \to \mathbb{R}^2$ given by

$$H(a, b) = (1, f(a, b))$$

is called the direction field. A drawing of the vector field H at a large number of points of D gives us approximate solution curves.

Along the curves $f(x, y) = c$, where c is a constant, the slopes are constant. These curves are called isoclines.

Example Revisited

Find the curve through the point $(1, 1)$ in the xy-plane having the slope $-\frac{y}{x}$ at each of its points.
The relevant ODE is

$$y' = -\frac{y}{x}.$$

By inspection,

$$y = \frac{c}{x}$$

is its general solution for an arbitrary constant c; that is, a family of hyperbolas.
The initial condition given is

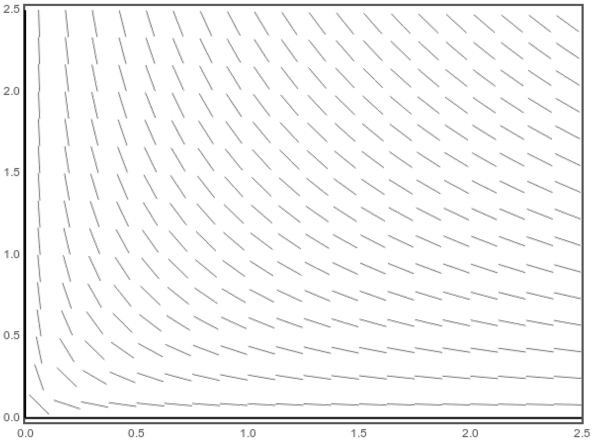
$$y(1) = 1,$$

which implies $c = 1$. Hence the particular solution for the above problem is

$$y = \frac{1}{x}.$$

Slope field

Direction field for $\frac{dy}{dx} = -\frac{y}{x}$.



Examples

A first order IVP can have

1. NO solution : $|y'| + |y| = 0, \ y(0) = 3.$
2. Precisely one solution : $y' = x, \ y(0) = 1.$ What is the solution?
3. Infinitely many solutions: $xy' = y - 1, \ y(0) = 1.$ The solutions are $y = 1 + cx$.

Motivation to study conditions under which the solution would exist and the conditions under which it will be unique!

Definition (Separable ODE)

An ODE of the form

$$M(x) + N(y)y^1 = 0$$

is called a separable ODE.

Method of separation of variables doesn't yield all solutions!

Solve $y' = 3y^{2/3},\ y(0) = 0.$
$y \equiv 0$ is a solution.
If $y \neq 0,\ \frac{dy}{y^{2/3}} = 3dx \implies 3y^{1/3} = 3(x + c) \implies y = (x + c)^3.$
Initial condition yields $c = 0$.
Hence $y = x^3$ and $y = 0$ are solutions which satisfy the initial conditions.
Consider

$$\phi_k(x) = \begin{cases} 0 & -\infty < x \leq k \\ (x - k)^3 & k < x < \infty \end{cases}$$

Are these functions solutions of the DE? YES.
There are infinitely many functions which are solutions of the DE.
Remark : This equation is non-linear - Motivation for existence/ uniqueness theorems.

Homogeneous ODE's

Definition

The first order ODE

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

is called homogeneous if M and N are homogeneous of equal degree.

What is important for the above method to work is that the ODE can be put into the form

$$y' = f(\frac{y}{x}).$$

Recall

So far, we saw how to solve separable ODE's and ODE's which can be put into the form

$$y^1 = f(\frac{y}{x}).$$

If the ODE is separable, just integration is enough to solve it. The homogeneous ODE's can be converted to separable ODE's by the substitution $y = vx$. Today, we will look at another class of first order ODE's - exact ODE's.

Exact ODE's

Definition

A first order ODE $M(x, y) + N(x, y)y^1 = 0$ is called exact if there is a function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = M \ \& \ \frac{\partial u}{\partial y} = N.$$

Recall from calculus that given a function $u(x, y)$ with continuous first partial derivatives, its differential is

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

If the ODE given above is exact, then the differential form

$$M(x, y)dx + N(x, y)dy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = du = 0.$$

Integrating $du = 0$, we get $u(x, y) = c$ as an implicit solution to the given ODE.

Solving Exact ODE's

Given an exact ODE as above, the function $u(x, y)$ can be found either by inspection or by the following method:
Step I: Integrate $\frac{\partial u}{\partial x} = M(x, y)$ with respect to x to get

$$u(x, y) = \int M(x, y)dx + k(y),$$

where $k(y)$ is a constant of integration.
Step II: To determine $k(y)$ differentiate the above equation in Step I with respect to y, to get:

$$\frac{\partial u}{\partial y} = k'(y) + \frac{\partial}{\partial y}\left(\int M(x, y)dx\right).$$

As the given ODE is exact, we get

$$N(x, y) = k'(y) + \frac{\partial}{\partial y}\left(\int M(x, y)dx\right).$$

We use this to determine $k(y)$ and hence u.

Example

Example: Solve the ODE:

$$(2x + y^2) + 2xy\frac{dy}{dx} = 0.$$

Consider the function $u(x, y) = x^2 + xy^2$. Note that

$$\frac{\partial u}{\partial x} = 2x + y^2, \ \frac{\partial u}{\partial y} = 2xy.$$

Hence $x^2 + xy^2 = c$ is an implicit general solution of the given ODE.

Closed Forms

Definition

The differential form

$$M(x, y)dx + N(x, y)dy$$

is called closed if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

i.e.,

$$M_y = N_x.$$

Does this definition remind you of anything from MA 105?