GRADIENTS & TANGENT PLANES

SUPPOSE $f: U \to \mathbb{R}$ ($U \subseteq \mathbb{R}^3$) AND $(x_0, y_0, z_0) \in U$ IS INTERIOR.

IF EACH OF f_x, f_y, f_z EXIST AT (x_0, y_0, z_0) WE DEFINE THE GRADIENT OF f AT (x_0, y_0, z_0) TO BE

$$\nabla f(x_0, y_0, z_0) := \left(f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0)\right)$$

SUPPOSE f_x, f_y, f_z ARE CONTINUOUS AT (x_0, y_0, z_0). FOR ANY UNIT VECTOR $\vec{u}$ ($\|\vec{u}\| = 1$)

$$(D_u f)(x_0, y_0, z_0) = \left\langle \nabla f(x_0, y_0, z_0), \vec{u} \right\rangle$$

SUPPOSE ANY 2 POINTS OF U CAN BE JOINED BY A PIECEWISE LINEAR PATH WITH EACH PIECE PARALLEL TO ONE OF THE AXES. SUPPOSE $\nabla f(x, y, z) = 0 \ \forall \ (x, y, z) \in U$ THEN $f \equiv$ CONST. ON U.

🚩 IF ONE OF f_x, f_y, f_z IS NOT CONTINUOUS AT (x_0, y_0, z_0), THEN

$$(D_u f)(x_0, y_0, z_0) = \langle \nabla f(x_0, y_0, z_0), \vec{u} \rangle$$

MAY NOT HOLD.

$$f(x,y) = \frac{x^3}{x^2+y^2} \quad \text{IF } (x,y) \neq (0,0)$$
$$= 0 \quad \text{IF } (x,y) = (0,0)$$

$$\nabla f(0,0) = (f_x(0,0), f_y(0,0)) =$$

$$(D_{\vec{u}} f)(0,0) =$$

$$\frac{\partial}{\partial x}\left(\frac{x^3}{x^2+y^2}\right) = \frac{3x^2(x^2+y^2) - x^3(2x)}{(x^2+y^2)^2} = \frac{x^4+3x^2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial y}\left(\frac{x^3}{x^2+y^2}\right) = \frac{x^3(-2y)}{(x^2+y^2)^2} = \frac{-2x^3y}{(x^2+y^2)^2}$$

FOR u_1, u_2, $D_u f(0,0) = \lim_{t \to 0} \frac{f(tu_1, tu_2)}{t} = \frac{t^3 u_1^3}{t^3(u_1^2+u_2^2)} = \boxed{u_1^3}$

$(u_1^2 + u_2^2 = 1$, AS (u_1, u_2) IS A UNIT VECTOR$)$

CHECK THAT $D_u f \neq \langle \nabla f(0,0), (u_1, u_2) \rangle$. (EXERCISE)

LET $f: D \to \mathbb{R}$ BE DIFFERENTIABLE AT (x_0, y_0, z_0). AND $\nabla f(x_0, y_0, z_0) \neq (0,0,0)$.

LET $\vec{u}$ BE A UNIT VECTOR. THEN

- THE DIRECTION IN WHICH f INCREASES MOST RAPIDLY IS $\nabla f(x_0, y_0, z_0)$.
- THE DIRECTION IN WHICH f DECREASES MOST RAPIDLY IS $-\nabla f(x_0, y_0, z_0)$.
- IF $\vec{v}$ IS SUCH THAT $\langle \nabla f(x_0, y_0, z_0), \vec{v} \rangle = 0$ THEN f DOES NOT CHANGE ALONG $\vec{v}$.

THIS FOLLOWS FROM $D_u f = \langle \nabla f, \vec{u} \rangle$ AND MAXIMIZING OR MINIMIZING THE ANGLE BETWEEN ∇f AND $\vec{u}$.

INDEED, $\langle \nabla f, \vec{u} \rangle = \|\nabla f\| \cdot \|\vec{u}\| \cos\theta$.

SINCE $\|\vec{u}\| = 1$, AND ∇f IS FIXED, $\langle \nabla f, \vec{u} \rangle$ IS DETERMINED BY $\cos\theta$ (EQUIVALENTLY, θ).

$$\theta = 0 \Rightarrow \cos\theta = 1 \Rightarrow \langle \nabla f, \vec{u} \rangle \uparrow \text{MAX.}$$

$$\theta = \pi \Rightarrow \cos\theta = -1 \Rightarrow \langle \nabla f, \vec{u} \rangle \downarrow \text{MAX}$$

TANGENT & NORMAL

SUPPOSE $U \subseteq \mathbb{R}^3$, $F: U \to \mathbb{R}$ IS DIFFERENTIABLE.

LET

$$S_\alpha = \{(x,y,z) \in U \mid F(x,y,z) = \alpha\}$$

LET $P = (x_0, y_0, z_0) \in S_\alpha$, AND C IS ANY SMOOTH CURVE ON S_α CONTAINING P.

THEN

$\langle \nabla F(P), \tau \rangle = 0$, WHERE τ IS THE TANGENT VECTOR TO C AT P.

SUPPOSE $\nabla F(P) \neq (0,0,0)$. THE VECTOR $\nabla F(P)$ IS CALLED THE NORMAL TO S_α AT P.

THE TANGENT PLANE TO S_α AT P IS THE PLANE

$$F_x(P)(x - x_0) + F_y(P)(y - y_0) + F_z(P)(z - z_0) = 0$$

THE LINE GIVEN BY

$$\frac{x - x_0}{F_x(P)} = \frac{y - y_0}{F_y(P)} = \frac{z - z_0}{F_z(P)}$$ IS CALLED THE

NORMAL LINE TO S_α AT P.

EXAMPLE

$$z^2 = 2x^2 - 2y^2 + 4$$

CONSIDER $(x_0, y_0, z_0) = (1,1,2)$ ON THIS SURFACE.

$$2y^2 - 2x^2 + z^2 = 4$$

$F(x,y,z) = 2y^2 - 2x^2 + z^2$; $\alpha = 4$, SO GIVEN SURFACE IS S_α.

$$\nabla F = (-4x, 4y, 2z)\Big|_{(1,1,2)} = (-4, 4, 4)$$

SO TANGENT PLANE TO S_α AT $(1,1,2)$ IS

$$-(x-1) + (y-1) + (z-2) = 0$$

$$\Leftrightarrow \quad -x + y + z = 2.$$

AND THE NORMAL LINE TO S_α AT $(1,1,2)$ IS GIVEN BY

$$\frac{x-1}{-1} = \frac{y-1}{1} = \frac{z-2}{1} \quad (= t)$$

2ND ORDER PARTIAL DERIVATIVES

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} := \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} := \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} := \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \quad \left(f_{xy} = (f_x)_y\right)$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} := \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

SUPPOSE $(x_0,y_0) \in U \subseteq \mathbb{R}^2$, $f : B_r(x_0,y_0) \to \mathbb{R}$ SUCH THAT f_x, f_y, f_{xy}, f_{yx} ARE ALL CONTINUOUS AT (x_0,y_0). THEN

$$f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0).$$

THE CONTINUITY HYPOTHESIS CANNOT BE DROPPED.

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{IF } (x,y) \neq (0,0) \\ 0 & \text{IF } (x,y) = (0,0). \end{cases}$$

CHECK THAT $f_{xy}(0,0) \neq f_{yx}(0,0)$. (EXERCISE)

MAXIMA / MINIMA

$f : U \to \mathbb{R}$ $(U \subseteq \mathbb{R}^2)$. LET $(x_0, y_0) \in U$ (INTERIOR).

(x_0, y_0) IS A POINT OF LOCAL MINIMUM

IF THERE EXISTS $\delta > 0$ S.T

$$f(x,y) \geq f(x_0,y_0) \quad \forall (x,y) \in B_\delta(x_0,y_0).$$

(x_0, y_0) IS A POINT OF LOCAL MAXIMUM

IF $f(x,y) \leq f(x_0,y_0) \quad \forall (x,y) \in B_\delta(x_0,y_0)$

FOR SOME $\delta > 0$.

IF K IS A CLOSED + BOUNDED SUBSET OF $\mathbb{R}^2$, $f : K \to \mathbb{R}$ IS CONTINUOUS, THEN WE KNOW THAT f IS BOUNDED AND ATTAINS ITS MAX/MIN.

QUESTION: HOW DO WE DETERMINE THE POINT(S) WHERE f ATTAINS MAX/MIN IN THE MULTIVARIATE CASE?