MA108 ODE: The Laplace Transform
Lecture 17 (D2)

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The Improper Integral of the first kind

Definition

A function $f : [a, b] \to \mathbb{R}$ is said to be piecewise continuous on $[a, b]$ if there is a partition

$$a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$$

such that

(i) f is continuous on (t_{i-1}, t_i) for $i = 1, 2, \ldots, n$.

(ii) $\lim\limits_{t \to t_i^+} f(t)$ and $\lim\limits_{t \to t_i^-} f(t)$ both exist for $i = 1, 2, \ldots, n-1$ and $\lim\limits_{t \to t_0^+} f(t)$ and $\lim\limits_{t \to t_n^-} f(t)$ both exist.

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- Let $f : [a, \infty) \to \mathbb{R}$ be a function. If f is such that, for any $b \geq a$, $f : [a, b] \to \mathbb{R}$ is piecewise continuous, then we say that f is piecewise continuous on $[a, \infty)$.
- Note that such an f is bounded on $[a, b]$ for every $b \geq a$.
- Note that, for f as above, the usual Riemann integral

$$I(b) = \int_a^b f(x)\, dx$$

exists for any $b \geq a$.

Definition

An improper integral of first kind is defined to be

$$\int_a^\infty f(x)\, dx := \lim_{b \to \infty} \int_a^b f(x)\, dx,$$

if this limit exists.

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If the above limit exists, we say that $\int_a^\infty f(x)\,dx$ <u>converges</u>, otherwise it is said to <u>diverge</u>.

Example: Consider the improper integral $\int_1^\infty \frac{dx}{x^s}$ for $s \in \mathbb{R}$.

For $s \neq 1$, consider $I(b) = \int_1^b \frac{dx}{x^s} = \frac{b^{1-s}-1}{1-s}$.

$$I(b) = \begin{cases} \frac{b^{1-s}-1}{1-s} & \text{if} \quad s \neq 1, \\ \ln b & \text{if} \quad s = 1. \end{cases}$$

So that

$$\lim_{b\to\infty} I(b) = \begin{cases} \frac{1}{s-1} & \text{if} \quad s > 1, \\ \infty & \text{if} \quad s \leq 1. \end{cases}$$

The Improper Integral of the first kind

Example: Consider the improper integral $\int_0^\infty \sin x \, dx$. Consider

$I(b) = \int_0^b \sin x \, dx = 1 - \cos b$.

Since $\lim_{b\to\infty} I(b)$ does not exist, the integral $\int_0^\infty \sin x \, dx$ diverges.

The Improper Integral of the first kind

Note that we can define similarily

$$\int_{-\infty}^{b} f(x)\, dx = \lim_{a \to -\infty} \int_{a}^{b} f(x)\, dx,$$

provided the limit exists.

Definition

The integral $\int_{-\infty}^{\infty} f(x)\, dx$ is said to be convergent if there is a $c \in \mathbb{R}$ such that $\int_{-\infty}^{c} f(x)\, dx$ is convergent and $\int_{c}^{\infty} f(x)\, dx$ is convergent. We then define

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{c} f(x)\, dx + \int_{c}^{\infty} f(x)\, dx.$$

Exercise: Show that the above definition is independent of the choice of c.

Theorem (Convergence Tests for Improper Integral)

Suppose there is a real number $M > 0$ such that

$$\int_a^b |f(x)|\, dx \leq M$$

for every $b \geq a$. Then $\int_a^\infty f(x)\, dx$ and $\int_a^\infty |f(x)|\, dx$ are convergent.

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Theorem (Comparison Test)

Suppose $0 \leq f(x) \leq g(x)$ *for every* $x \geq a$. *If* $\int_a^\infty g(x)\, dx$ *converges, then* $\int_a^\infty f(x)\, dx$ *also converges and*

$$\int_a^\infty f(x)\, dx \leq \int_a^\infty g(x)\, dx.$$

Example: As

$$0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$$

on $[1, \infty)$, and $\int_1^\infty \frac{1}{x^2}\, dx$ converges, it follows that $\int_1^\infty \frac{\sin^2 x}{x^2}\, dx$ also converges.

The Improper Integral of the first kind

Theorem (Limit Comparison Test)

Suppose $f(x) \geq 0$ and $g(x) > 0$ on $[a, \infty)$. Suppose that $\int_a^b f(x)\, dx$ and $\int_a^b g(x)\, dx$ exist for every $b \geq a$ and

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = c$$

(i) *If $c \neq 0$, then either both $\int_a^\infty f(x)\, dx$ and $\int_a^\infty g(x)\, dx$ converge or diverge.*

(ii) *If $c = 0$, and $\int_a^\infty g(x)\, dx$ converges, then $\int_a^\infty f(x)\, dx$ converges.*

Example: Consider $\int_1^\infty e^{-x} x^s\, dx$ for $s \in \mathbb{R}$.
Note that

$$\lim_{x \to \infty} \frac{e^{-x} x^s}{x^{-2}} = 0$$

and $\int_1^\infty \frac{dx}{x^2}$ converges. Hence, by the above theorem $\int_1^\infty e^{-x} x^s dx$ converges for every $s \in \mathbb{R}$.

Example: Define the Gamma function $\Gamma : (0, \infty) \to \mathbb{R}$ is defined by

$$\Gamma(y) = \int_0^\infty e^{-x} x^{y-1} dx.$$

How do we know that the right hand side integral converges? Can write it as

$$\int_0^1 e^{-x} x^{y-1} dx + \int_1^\infty e^{-x} x^{y-1} dx,$$

and we need to check that both these integrals do converge. Why do these integrals converge?
Hint: For the first, $e^{-x} x^{y-1} \leq x^{y-1}$. Thus, the first integral $\leq \frac{1}{y}$, and thus converges.

Hence, $\Gamma(y)$ is well-defined for $y > 0$.

Gamma Function

The gamma function satisfies a nice functional equation:

$$\Gamma(y+1) = y\Gamma(y).$$

Proof: Let $0 < a < b$. Use integration by parts to see:

$$\begin{aligned}\int_a^b e^{-x}x^y\,dx &= [-x^y e^{-x}]_a^b + y\int_a^b e^{-x}x^{y-1}dx \\ &= a^y e^{-a} - b^y e^{-b} + y\int_a^b e^{-x}x^{y-1}dx.\end{aligned}$$

Take limit as $b \to \infty$ and $a \to 0^+$ to get

$$\int_0^\infty e^{-x}x^y\,dx = y\Gamma(y),$$

i.e., $\Gamma(y+1) = y\Gamma(y)$. In particular, for $n = 1, 2, \ldots$

$$\Gamma(n+1) = n!.$$

Check! Use Induction to verify the above equation.

Thus, the gamma function interpolates the factorial function.

The Improper Integral of the second kind

Let $f : (a, b] \to \mathbb{R}$ be such that $\int_x^b f(t)\,dt$ exists for every $x \in (a, b]$. For $x \in (a, b]$, set

$$I(x) = \int_x^b f(t)\,dt$$

Definition

If $\lim_{x \to a^+} I(x)$ exists, then we say that $\int_a^b f(t)\,dt$ is convergent and call it an improper integral of second kind.

Similarily, we can define improper integrals of second kind for $f : [a, b) \to \mathbb{R}$.

Example: Let $f : (0, \infty) \to \mathbb{R}$ be defined by

$$f(t) = \frac{1}{t^s}$$

for $s \in \mathbb{R}$. For $b, x > 0$, consider

$$I(x) = \int_x^b \frac{dt}{t^s} = \begin{cases} \dfrac{b^{1-s} - x^{1-s}}{1-s} & \text{if} \quad s \neq 1, \\ \ln b - \ln x & \text{if} \quad s = 1. \end{cases}$$

Hence, $\displaystyle\lim_{x \to 0^+} \int_x^b \frac{dt}{t^s}$ exists if and only if $s < 1$.

The Improper Integral of the second kind

Definition

Let $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$ be a partition of $[a, b]$ and let $f : [a, b] \setminus \{t_1, t_2, \ldots, t_n\} \to \mathbb{R}$. We say that $\int_a^b f(t)\, dt$ converges if each of the integrals $\int_{t_{i-1}}^{t_i} f(t)\, dt, i = 1, 2, \ldots, n$ converge, and

$$\int_a^b f(t)\, dt = \int_a^{t_1} f(t)\, dt + \int_{t_1}^{t_2} f(t)\, dt + \ldots + \int_{t_n}^{b} f(t)\, dt.$$

Example: Consider $\displaystyle\int_0^3 \frac{dx}{(x-1)^{2/3}}$.

The function $(x-1)^{-2/3}$ is not defined at $x = 1$. Note that

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = 3 \text{ and } \int_1^3 \frac{dx}{(x-1)^{2/3}} = 3\sqrt[3]{2}.$$

Hence,

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}} = 3 + 3\sqrt[3]{2}.$$

Example

Exercise: Prove that

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

Note that e^{-x^2} is continuous for every x and $\int_0^1 e^{-x^2} dx$ is a proper integral. We need to check that the improper integral $\int_1^\infty e^{-x^2} dx$ converges. To see this, note that

$$\int_1^\infty e^{-x^2} dx \leq \int_1^\infty e^{-x} dx = \frac{1}{e}.$$

Hence, $\int_1^\infty e^{-x^2} dx$ converges. To find it's value, note that

$$I = \int_0^\infty e^{-x^2} dx = \int_0^\infty e^{-y^2} dy$$

so that

$$I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy.$$

Hint: Compute I^2 as a double integral by changing to polar coordinates.

Example

Exercise: Find $\Gamma(\frac{1}{2}), \Gamma(\frac{3}{2}), \Gamma(\frac{5}{2}), \ldots$.
By definition,

$$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-x} x^{-\frac{1}{2}} dx.$$

Put $x = t^2$. Thus,

$$\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-t^2} dt = 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}.$$

Now,

$$\Gamma(\frac{3}{2}) = \Gamma(\frac{1}{2}+1) = \frac{1}{2} \cdot \Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}.$$

Similarily,

$$\Gamma(\frac{5}{2}) = \Gamma(\frac{3}{2}+1) = \frac{3}{2} \cdot \Gamma(\frac{3}{2}) = 3\frac{\sqrt{\pi}}{4}.$$