MA108 ODE: Picard's Theorem
Lecture 5 (D2)

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Existence and Uniqueness

Theorem

(i) *If f is continuous on an open rectangle*

$$R = \{(x, y) \in \mathbb{R}^2 : a < x < b, c < y < d\}$$

that contains the point (x_0, y_0), then the IVP

$$y' = f(x, y), y(x_0) = y_0 \tag{1}$$

has at least one solution on some open subinterval of (a, b) that contains x_0.

(ii) *If both f and f_y are continuous on R, then (1) has a unique solution on some open subinterval of (a, b) that contains x_0.*

Example: Consider the IVP

$$y' = \frac{10}{3} x y^{2/5}, y(x_0) = y_0.$$

(i) For what points (x_0, y_0), does the Theorem imply that it has a solution?

(ii) For what points (x_0, y_0), does the Theorem imply that it has a unique solution on some open interval that conatins x_0?

Since $f(x, y) = \frac{10}{3} x y^{2/5}$ is continuous for all (x, y), it follows that the above IVP has a solution for every (x_0, y_0). Here

$$f_y(x, y) = \frac{4}{3} x y^{-3/5}$$

is continuous for all (x, y) with $y \neq 0$. Therefore, if $y_0 \neq 0$, there is an open rectangle on which both f and f_y are continuous and hence the above IVP has a unique solution on some interval that contains x_0. If $y = 0$, then $f_y(x, y)$ is undefined. Hence, the Theorem does not apply to this IVP if $y_0 = 0$.

Example: Consider the IVP

$$y' = \frac{10}{3}xy^{2/5}, y(0) = -1.$$

This IVP has a unique solution on some open interval that contains $x_0 = 0$. Find a solution and determine the largest open interval (a, b) on which it is unique.

Let y be any solution of the above IVP. Since $y(0) = -1$, it follows from the continuity of y that there is an open interval I that contains $x_0 = 0$ on which y has no zeroes. Separating the variables, we get

$$y^{-2/5}y' = \frac{10}{3}x.$$

Integrating this and writing the arbitrary constant as $5c/3$, we get

$$\frac{5}{3}y^{3/5} = \frac{5}{3}(x^2 + c).$$

Example Continued

$$\frac{5}{3}y^{3/5} = \frac{5}{3}(x^2 + c).$$

Since $y(0) = -1$, $c = -1$ so

$$y = (x^2 - 1)^{5/3}$$

for $x \in I$. This is a unique solution to the IVP on $(-1, 1)$. This is the largest open interval on which the given IVP has a unique solution. To see this, note that

$$y = (x^2 - 1)^{5/3}$$

is a solution of the given IVP on $(-\infty, \infty)$. There are infinitely many solutions of the given IVP that differ from $y = (x^2 - 1)^{5/3}$ on every open interval larger than $(-1, 1)$. One such solution is

$$y(x) = \begin{cases} (x^2 - 1)^{5/3} & -1 < x < 1 \\ 0 & |x| \geq 1 \end{cases}$$

Corollary

Consider the IVP

$$y^1 + p(t)y = g(t);\ y(t_0) = y_0,$$

where p and g are continuous functions on an interval I with $t_0 \in I$. Then there is a unique solution on I of the given IVP.

Proof.

Since $y^1 = -p(t)y - g(t)$, it follows that

$$f(t,y) = -p(t)y - g(t) \text{ and } \frac{\partial f}{\partial y}(t,y) = -p(t)$$

are both continuous on $I \times \mathbb{R}$. By the existence and uniqueness theorem, the given IVP has a unique solution on a subinterval $J \subseteq I$ with $t_0 \in J$.

$\square$

Picard's Iteration Method

Picard's iteration method gives us a rough idea on how to construct solutions to IVP's. Consider the IVP

$$y^1 = f(t, y);\; y(0) = 0.$$

Suppose $y = \phi(t)$ is a solution to the IVP. Then,

$$\frac{d\phi}{dt} = f(t, \phi(t)),\; \phi(0) = 0.$$

That is,

$$\phi(t) = \int_0^t f(s, \phi(s))ds;\; \phi(0) = 0.$$

The above equation is called an integral equation in the unknown function ϕ.

Picard's Iteration Method

Conversely, if the integral equation holds i.e.,

$$\phi(t) = \int_0^t f(s, \phi(s))ds; \ \phi(0) = 0,$$

then by the Fundamental Theorem of Calculus,

$$\frac{d\phi}{dt} = f(t, \phi(t)),$$

so that $y = \phi(t)$ is a solution to the IVP $y^1 = f(t, y); \ y(0) = 0$. Thus, solving the integral equation is equivalent to solving the IVP.

Picard's Iteration Method

Picard's iteration describes a way to look for solutions of the integral equation

$$\phi(t) = \int_0^t f(s, \phi(s))ds.$$

We define iteratively a sequence of functions $\phi_n(t)$ for every integer $n \geq 0$ as follows: Let

$$\begin{aligned}
\phi_0(t) &\equiv 0 \\
\phi_1(t) &= \int_0^t f(s, \phi_0(s))ds \\
\phi_2(t) &= \int_0^t f(s, \phi_1(s))ds \\
&\cdot \\
&\cdot \\
\phi_{n+1}(t) &= \int_0^t f(s, \phi_n(s))ds.
\end{aligned}$$

Picard's Iteration Method

Note: Each ϕ_n satisfies the initial condition $\phi_n(0) = 0$. None of the ϕ_n may satisfy $y^1 = f(t, y)$. Suppose for some n, $\phi_{n+1} = \phi_n$. Then,

$$\phi_{n+1} = \phi_n = \int_0^t f(s, \phi_n(s))ds,$$

and this implies

$$\frac{d}{dt}(\phi_n(t)) = f(t, \phi_n(t))$$

is a solution of the given IVP. In general, the sequence $\{\phi_n\}$ may not terminate. In fact, all the ϕ_n may not even be defined outside a small region in the domain. However, it is possible to show that, under the hypotheses of the above theorem, the sequence converges to a function

$$\phi(t) = \lim_{n\to\infty} \phi_n(t)$$

which is the unique solution to the given IVP.

Example

Example: Solve the IVP:

$$y^1 = 2t(1+y);\ \ y(0) = 0.$$

The corresponding integral equation is

$$\phi(t) = \int_0^t 2s(1+\phi(s))ds.$$

Let $\phi_0(t) \equiv 0$. Then,

$$\begin{aligned}
\phi_1(t) &= \int_0^t 2sds = t^2, \\
\phi_2(t) &= \int_0^t 2s(1+s^2)ds = t^2 + \frac{t^4}{2}, \\
\phi_3(t) &= \int_0^t 2s(1+s^2+\frac{s^4}{2})ds = t^2 + \frac{t^4}{2} + \frac{t^6}{6}.
\end{aligned}$$

Example continued

We claim:

$$\phi_n(t) = t^2 + \frac{t^4}{2} + \frac{t^6}{6} + \ldots + \frac{t^{2n}}{n!}.$$

Use induction to prove this:

$$\begin{aligned}
\phi_{n+1}(t) &= \int_0^t 2s(1+\phi_n(s))ds \\
&= \int_0^t 2s\left(1 + s^2 + \frac{s^4}{2} + \ldots + \frac{s^{2n}}{n!}\right) ds \\
&= t^2 + \frac{t^4}{2} + \frac{t^6}{6} + \ldots + \frac{t^{2n}}{n!} + \frac{t^{2n+2}}{(n+1)!}.
\end{aligned}$$

Hence $\phi_n(t)$ is the n-th partial sum of the series $\displaystyle\sum_{k=1}^{\infty} \frac{t^{2k}}{k!}$.

Example continued

Recall that $\phi_n(t)$ is the n-th partial sum of the series $\sum_{k=1}^{\infty} \frac{t^{2k}}{k!}$.

Applying the ratio test, we get:

$$\left| \frac{t^{2k+2}}{(k+1)!} \cdot \frac{k!}{t^{2k}} \right| = \frac{t^2}{k+1} \to 0$$

for all t as $k \to \infty$. Thus,

$$\lim_{n\to\infty} \phi_n(t) = \sum_{k=1}^{\infty} \frac{t^{2k}}{k!} = e^{t^2} - 1.$$

Hence, $y(t) = e^{t^2} - 1$ is a solution of the IVP.