CONTINUITY

🚩 SUPPOSE $f:[a,b] \to \mathbb{R}$. LET $c \in [a,b]$

WE SAY f IS CONTINUOUS AT c

IF

$$\lim_{x \to c} f(x) \text{ EXISTS AND}$$

$$\lim_{x \to c} f(x) = f(c)$$

🚩 WE SAY f IS CONTINUOUS ON $[a,b]$ IF

f IS CONTINUOUS AT EACH $c \in [a,b]$.

🚩 IF $c = a$, THEN WE ONLY CONSIDER

$$\lim_{x \to a^+} f(x);$$

IF $c = b$ WE ONLY CONSIDER

$$\lim_{x \to b^-} f(x)$$

SUPPOSE $f, g : [a,b] \to \mathbb{R}$

🚩 IF f, g ARE CONTINUOUS AT $c \in [a,b]$

THEN $f \pm g$ IS CONTINUOUS AT c.

🚩 $\alpha \in \mathbb{R}$, αf IS CONTINUOUS AT c.

🚩 fg IS CONTINUOUS AT c

🚩 IF $g(c) \neq 0$ THEN $\frac{f}{g}$ IS CONTINUOUS

AT c.

EXAMPLES

POLYNOMIAL FUNCTIONS

$$p : \mathbb{R} \to \mathbb{R}$$

$$p(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$$

WITH $a_0, a_1, \cdots, a_n \in \mathbb{R}$

ARE CONTINUOUS.

🚩 $$f(x) = 1 \text{ if } x \in \mathbb{Q}$$
$$= 0 \text{ if } x \in \mathbb{R} \setminus \mathbb{Q}$$

THIS IS NOWHERE CONTINUOUS

🚩 IF f IS CONTINOUS AT c AND g IS CONTINUOUS AT $f(c)$, THEN $g \circ f$ IS CONTINUOUS AT c.

🚩 THE TRIGONOMETRIC FUNCTIONS, $\sin : \mathbb{R} \to \mathbb{R}$, $\cos : \mathbb{R} \to \mathbb{R}$ ARE BOTH CONTINUOUS*.

AND BY VIRTUE OF THE PROPERTIES LISTED EARLIER, ALL TRIGONOMETRIC FUNCTIONS ARE CONTINUOUS ON THEIR DOMAINS.

* TO SHOW THAT sin IS CONTINUOUS, WE USE

$$|\sin x - \sin a| = \left|2 \cos\left(\frac{x+a}{2}\right) \sin\left(\frac{x-a}{2}\right)\right|$$

SO IT SUFFICES TO SHOW

$$\lim_{x \to 0} \sin x = 0.$$

A SIMILAR CALCULATION WORKS FOR $\cos x$ AS WELL.

DISCONTINUITIES

f IS DISCONTINUOUS AT c IF IT IS <u>NOT</u> CONTINUOUS AT c.

ESSENTIAL DISCONTINUITY

$$\lim_{x \to c^+} f(x) \quad \text{OR} \quad \lim_{x \to c^-} f(x)$$

(OR BOTH)

DOES NOT EXIST.

$$\boxed{\begin{aligned} f(x) &= \frac{1}{x}, & x &\neq 0 \\ &= 0 & x &= 0 \end{aligned}}$$

JUMP DISCONTINUITY

BOTH $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$ EXIST

BUT

$$\lim_{x \to c^+} f(x) \neq \lim_{x \to c^-} f(x)$$

$(f(x) = \lfloor x \rfloor)$

REMOVABLE DISCONTINUITY

$\lim_{x \to c} f(x)$ EXISTS BUT

$$\lim_{x \to c} f(x) \neq f(c)$$

EXPONENTIAL FUNCTION

WHAT IS 2^x ? (x IS ARBITRARY REAL)

$x \in \mathbb{N}$,

$$2^0 := 1$$
$$2^{n+1} := (2^n)2 \quad (\text{INDUCTIVE FOR } x \in \mathbb{N})$$

$x \in \mathbb{Z}$. IF $x = -n$, $n \in \mathbb{N}$

$$2^{-n} := \frac{1}{2^n}$$

EASY TO CHECK: $2^{x+y} = 2^x \cdot 2^y$

$$\boxed{2^{\frac{1}{100}} = ?}$$ AN x S.T $x^{100} = 2$

BY INTERMEDIATE VALUE THEOREM, SUCH AN x EXISTS

$2^{1/n}$ EXISTS FOR ALL $n \geq 2$

$$2^{m/n} := \left(2^{1/n}\right)^m. \quad 2^{\sqrt{2}} = ?$$

TAKE $x_n \to \sqrt{2}$, $x_n \in \mathbb{Q}$.

$$2^{x_n} \text{ IS DEFINED.} \quad 2^{\sqrt{2}} := \lim_{n\to\infty} 2^{x_n}$$

IT CAN BE SHOWN THAT THIS LIMIT IS INDEPENDENT OF THE SEQUENCE (x_n).

3 IMPORTANT PROPERTIES

🚩 SUPPOSE $f : [a,b] \to \mathbb{R}$ IS CONTINUOUS.

THEN THERE EXIST $m, M^* \in \mathbb{R}$ S.T.

(i) $m \leq f(x) \leq M$ FOR ALL $x \in [a,b]$

(ii) THERE EXIST $x_0, y_0 \in [a,b]$ S.T.

(* m, M DEPEND ON f) $f(x_0) = m, \quad f(y_0) = M$

IS THIS TRUE IF WE REPLACE $[a,b]$ BY (a,b) ?

NO! $(a,b) = (0,1)$

(i) $f(x) = x$

$m = 0, \quad M = 1, \qquad 0 \leq f(x) \leq 1,$ BUT NEITHER EQUALITY ATTAINED

(ii) $(a,b) = (0,1), \quad f(x) = \frac{1}{x}, \quad f$ IS CONTINUOUS ON $(0,1)$, BUT NOT BOUNDED

IF m, M ARE AS IN THE PRECEDING FACT, AND SUPPOSE

(MINIMUM OF f) $m \leq \alpha \leq M$. (MAX. OF f IN THAT INTERVAL)

THEN THERE EXISTS $x_0 \in [a,b]$ s.t.

$$f(x_0) = \alpha$$

THIS IS REFERRED TO AS

INTERMEDIATE VALUE PROPERTY.

SUPPOSE $f : [a,b] \to \mathbb{R}$ IS CONTINUOUS AND 1-1. THEN f IS STRICTLY INCREASING OR DECREASING. FURTHERMORE, IF

$\text{Ran}(f) = [c,d]$ (WHY?

THEN $f^{-1} : [c,d] \to [a,b]$ IS ALSO INCREASING OR DECREASING AS f IS, AND f^{-1} IS ALSO CONTINUOUS.

🚩 IF $f:[a,b] \to [a,b]$ IS CONTINUOUS, THEN $\exists\ c \in [a,b]$ s.t. $f(c) = c$.

🚩 FOR EACH $n \geq 2$, $n \in \mathbb{N}$, THERE EXISTS A CONTINUOUS FUNCTION DENOTED BY

$$f(x) = \sqrt[n]{x} \quad (n^{TH} \text{ ROOT OF } x)\ (x \geq 0)$$

WHICH SATISFIES,

$$(f(x))^n = x \quad \forall\ x \in \mathbb{R}.$$

$g(x) = x^n$ on $[0, \infty)$