MA108 ODE: Second Order Linear ODE's
Lecture 7 (D2)

Prachi Mahajan
IIT Bombay

Uniqueness Theorem

Theorem

Consider the IVP

$$y'' + p(x)y' + q(x)y = 0, y(x_0) = a, y'(x_0) = b,$$

where p and q are continuous on an interval I, x_0 is any point in I, and a, b are real numbers. Then there is a unique solution to the IVP on I.

Theorem

If y_1 and y_2 are solutions of homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \tag{1}$$

on an interval I, then any linear combination

$$y = c_1 y_1 + c_2 y_2$$

of y_1 and y_2 is also a solution of (1) on I.

(Why?).

Let

$$C(I) = \{f : I \to \mathbb{R} \mid f \text{ is continuous}\}$$
$$C^1(I) = \{f : I \to \mathbb{R} \mid f, f' \text{ are continuous}\}$$
$$C^2(I) = \{f : I \to \mathbb{R} \mid f, f', f'' \text{ are continuous}\}.$$

Check: $C(I), C^1(I), C^2(I)$ are vector spaces with addition and scalar multiplication defined as:

$$(f+g)(x) = f(x) + g(x), \; x \in I,$$

$$(k \cdot f)(x) = kf(x), \; k \in \mathbb{R}, x \in I.$$

Define

$$L : C^2(I) \to C(I)$$

by

$$L(f) = f'' + p(x)f' + q(x)f.$$

Check: L is a linear transformation; i.e.,

$$L(cf + dg) = cL(f) + dL(g),$$

for all $c, d \in \mathbb{R}$ and for all $f, g \in C^2(I)$.
Aside: Associated to any linear transformation, you would look at two important vector spaces. What are they?
The null space of L, denoted by $N(L)$ is

$$N(L) = \{f \in C^2(I) \mid L(f) = f'' + p(x)f' + q(x)f = 0\}.$$

Thus, $N(L)$ consists of solutions of the ODE

$$y'' + p(x)y' + q(x)y = 0.$$

Fundamental Theorem

Given a vector space, what's the most important thing about it? Dimension.

Theorem (Dimension Theorem)

Let I be an interval in $\mathbb{R}$, p, q be continuous on I and let

$$L : C^2(I) \to C(I)$$

be defined by

$$L(f) = f'' + p(x)f' + q(x)f$$

and the null space of L

$$N(L) = \{f \in C^2(I) \mid L(f) = f'' + p(x)f' + q(x)f = 0\}.$$

The dimension of $N(L) = 2 =$ order of the ODE.

Basis of Solutions

Given a vector space, after the dimension, what's the next most important thing? Basis. Dimension theorem told us that

$$\dim N(L) = 2.$$

Can we give an explicit basis?

Theorem

Let f, g be two solutions of the homogeneous second order linear ODE

$$y'' + p(x)y' + q(x)y = 0,$$

where p and q are continuous on an interval I in $\mathbb{R}$. Let $(f(x_0), f'(x_0))$ and $(g(x_0), g'(x_0))$ be linearly independent vectors in $\mathbb{R}^2$, for some $x_0 \in I$. Then the solution space is the linear span of f and g.

Proof. Let h be a solution of the given ODE. We want to find c and d such that

$$h(x) = cf(x) + dg(x).$$

Basis of Solutions

This implies

$$h(x_0) = cf(x_0) + dg(x_0)$$
$$h'(x_0) = cf'(x_0) + dg'(x_0).$$

Thus,

$$\begin{pmatrix} f(x_0) & g(x_0) \\ f'(x_0) & g'(x_0) \end{pmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} h(x_0) \\ h'(x_0) \end{bmatrix}.$$

As the column vectors

$$\begin{bmatrix} f(x_0) \\ f'(x_0) \end{bmatrix} \;\&\; \begin{bmatrix} g(x_0) \\ g'(x_0) \end{bmatrix}$$

are linearly independent, the matrix

$$W(x_0) = \begin{pmatrix} f(x_0) & g(x_0) \\ f'(x_0) & g'(x_0) \end{pmatrix}$$

is invertible.

Therefore,

$$c = \frac{\begin{vmatrix} h(x_0) & g(x_0) \\ h'(x_0) & g'(x_0) \end{vmatrix}}{\det W(x_0)},$$

and

$$d = \frac{\begin{vmatrix} f(x_0) & h(x_0) \\ f'(x_0) & h'(x_0) \end{vmatrix}}{\det W(x_0)}.$$

(What's this method called?) Let

$$u = h - cf - dg.$$

Then u is a solution of the IVP

$$y'' + p(x)y' + q(x)y = 0, y(x_0) = 0, y'(x_0) = 0,$$

which implies that $u \equiv 0$ by the uniqueness theorem. I.e.,

$$h(x) = cf(x) + dg(x)$$

on I.

Wronskian and Linear Independence

The dimension theorem says that if p and q are continuous in an interval I then the solutions of the ODE

$$y'' + p(x)y' + q(x)y = 0$$

form a vector space spanned by two linearly independent solutions. How to check for linear independence? We start with a definition.

Linearly independent & dependent functions - RECALL

Definition

The functions f and g are said to be linearly independent on an interval I if

$$c_1 f(x) + c_2 g(x) = 0 \;\; \forall x \in I \implies c_1 = c_2 = 0.$$

Examples : 1. The functions $\sin 2x$ and $\sin x \cos x$ are linearly dependent on $(-\infty, \infty)$.
2. The functions x and $|x|$ are linearly dependent on $(0, \infty)$ but are linearly independent on $(-\infty, \infty)$.

Definition

The Wronskian of any two differentiable functions f and g is defined by

$$W(f, g; x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}.$$

Wronskian and Linear Independence

Proposition

Suppose f and g are linearly dependent and differentiable on an interval I. Then, $W(f, g; x) = 0$ on I.

Proof. As f and g are linearly dependent, there exist $c, d \in \mathbb{R}$, not both 0, such that

$$cf(x) + dg(x) = 0.$$

Thus,

$$cf'(x) + dg'(x) = 0.$$

Hence,

$$\begin{pmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{pmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore, $W(f, g; x) = f(x)g'(x) - f'(x)g(x) = 0$ for all x (Why?) since $\begin{bmatrix} c \\ d \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Wronskian and Linear Independence

Note: The converse is not true. For instance, if $f(x) = x^2$ and

$$g(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0, \end{cases}$$

If $x \geq 0$, $W(x^2, x^2; x) = 0$. If $x < 0$, $W(x^2, -x^2; x) = 0$. Hence, $W(f, g; x) = 0$ for all $x \in \mathbb{R}$, but f and g are linearly independent.

Wronskian and Linear Independence

Theorem (Abel's Formula)

Let p, q be continuous on an interval I and let f, g be solutions of

$$y'' + p(x)y' + q(x)y = 0$$

on I. Let a be any point of I. Then

$$W(f,g;x) = W(f,g;a)e^{-\int_a^x p(t)dt}, x \in I$$

Proof. Set $W(f,g;x) = W(x)$. Then,

$$W(x) = (fg' - f'g)(x)$$
$$W'(x) = (fg'' - f''g)(x).$$

Wronskian and Linear Independence

Now,

$$f'' = -p(x)f' - q(x)f$$
$$g'' = -p(x)g' - q(x)g.$$

Thus,

$$\begin{aligned} W'(x) &= (fg'' - f''g)(x) \\ &= \left(-fpg' - fqg + gpf' + gqf\right)(x) \\ &= -p(x)\left(fg' - f'g\right)(x) \\ &= -p(x)W(x), \end{aligned}$$

i.e., W is the solution of the IVP

$$y' + p(x)y = 0, y(a) = W(a).$$

Hence,

$$W(x) = W(a)e^{-\int_a^x p(t)dt},$$

i.e.,

$$W(f,g;x) = W(f,g;a)e^{-\int_a^x p(t)dt}.$$

Wronskian and Linear Independence

Theorem

Let p, q be continuous on an interval I and let f, g be solutions of

$$y'' + p(x)y' + q(x)y = 0$$

on I. Then,

1. *If $W(f, g; a) = 0$ for some $a \in I$, then $W \equiv 0$ on I.*
2. *f and g are linearly dependent on I if and only if $W(f, g; a) = 0$ for some $a \in I$.*

Thus, f and g are linearly independent on I iff $W(f, g; x) \neq 0$ for all $x \in I$.

Proof of (1): Suppose $W(a) = 0$ for some $a \in I$. Then, for any $x \in I$,

$$W(x) = W(a)e^{-\int_a^x p(t)dt} = 0.$$

Hence, $W \equiv 0$ on I.

Wronskian and Linear Independence

Proof of (2): $\Rightarrow$ Done earlier. Need to do $\Leftarrow$.
Suppose that $W(f, g; a) = 0$ for some $a \in I$. If $f \equiv 0$ on I, then $\{f, g\}$ is linearly dependent. Suppose $f(x) \neq 0$ for some $x \in I$. There is a subinterval $J \subseteq I$ on which $f(x) \neq 0$ (Why?). On J, we have:

$$\left(\frac{g}{f}\right)'(x) = \left(\frac{fg' - f'g}{f^2}\right)(x) = \frac{W(f, g; x)}{f^2(x)} = 0.$$

(Why?) Thus, $\frac{g}{f} = k$, a constant on J. We want to show that $g(x) = kf(x)$ on I (How?) Note that, if $x_0 \in J$, then $Y = g - kf$ is a solution of the IVP

$$y'' + p(x)y' + q(x)y = 0, \; y(x_0) = 0, y'(x_0) = 0.$$

It follows from the Uniqueness Theorem that $Y \equiv 0$ on I, i.e., $g \equiv kf$ on I.