$\mathcal{P}'$ REFINES $\mathcal{P}$.

$$L(\mathcal{P}, f) \le L(\mathcal{P}', f) \le U(\mathcal{P}', f) \le U(\mathcal{P}, f)$$

SUPPOSE f IS MONOTONE ON $[a, b]$.

THEN f IS INTEGRABLE ON $[a, b]$

PROOF: IT SUFFICES TO CONSIDER EQUITABLE PARTITIONS, i.e., $x_{i+1} - x_i = x_i - x_{i-1}$ $\forall i$

SUPPOSE f IS $\uparrow$

$$L(\mathcal{P}, f) = (x_1 - x_0) f(x_0) + (x_2 - x_1) f(x_1) + \cdots + f(x_{n-1})(x_n - x_{n-1})$$

$$= \left(\frac{b-a}{n}\right)\left(f(x_0) + f(x_1) + \cdots + f(x_{n-1})\right)$$

SIMILARLY,

$$U(\mathcal{P}, f) = \left(\frac{b-a}{n}\right)\left[f(x_1) + f(x_2) + \cdots + f(x_n)\right]$$

$\Rightarrow$

$$U(\mathcal{P}, f) - L(\mathcal{P}, f) = \left(\frac{b-a}{n}\right)\left(f(b) - f(a)\right) < \epsilon$$

$(x_0 = a,\ x_n = b)$ IF n IS LARGE ENOUGH.