# MAXIMA / MINIMA

SUPPOSE $f: U \to \mathbb{R}$ AND $(x_0, y_0) \in U$ IS INTERIOR.

LET $\vec{u} = (u_1, u_2)$ BE A UNIT VECTOR. IF

$f$ HAS A LOCAL MAX/MIN. AT $(x_0, y_0)$

AND $(D_u f)(x_0, y_0)$ EXISTS, THEN $(D_u f)(x_0, y_0) = 0$

IN PARTICULAR, IF $f_x, f_y$ BOTH EXIST AT $(x_0, y_0)$

THEN $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$.

PROOF: IF $(x_0, y_0)$ IS A LOCAL MAX. THEN

$\exists\ \delta > 0$ S.T. $\forall\ (x, y) \in B_\delta(x_0, y_0)$

$f(x, y) \leq f(x_0, y_0) \Rightarrow f(x_0 + tu_1, y_0 + tu_2) \leq f(x_0, y_0)$

SINCE $D_u f(x_0, y_0)$ EXISTS,

$$\lim_{t \to 0^+} \frac{f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0)}{t} \leq 0$$

$$\lim_{t \to 0^-} \quad " \quad \geq 0$$

$$\Rightarrow \lim_{t \to 0} \frac{f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0)}{t} = 0$$

$(x_0, y_0)$ IS CALLED A CRITICAL POINT OF $f$

IF

(i) EITHER $f_x, f_y$ EXIST AND

$$f_x(x_0,y_0) = f_x(x_0,y_0) = 0$$

(ii) ONE OF $f_x(x_0,y_0)$, $f_y(x_0,y_0)$ DOES NOT EXIST. (AT LEAST ONE)

IF $f: K \to \mathbb{R}$ IS CONTINUOUS, ($K$ IS CLOSED AND BOUNDED) THEN $\max\limits_{(x,y)\in K} f(x,y)$ $\left(\text{resp. } \min\limits_{(x,y)\in K} f(x,y)\right)$ IS ATTAINED AT A CRITICAL POINT OF $K$, OR AT A BOUNDARY POINT OF $K$.

$f: U \to \mathbb{R}$ AND $P \in U$ IS A CRITICAL POINT.

WE SAY THAT $P$ IS A SADDLE POINT IF IN EACH $B_\delta(P) \cap U$, THERE EXIST POINTS $Q_1, Q_2 \in B_\delta(P) \cap U$ s.t

$$f(Q_1) < f(P) < f(Q_2).$$

# DETERMINING MAX/MIN

GIVEN $f: U \to \mathbb{R}$, AND A CRITICAL POINT $(x_0, y_0)$ FOR $f$.

HOW DO WE DETERMINE IF THIS IS A POINT OF LOCAL MAX. OR LOCAL MINIMA?

SECOND DERIVATIVE TEST.

SUPPOSE ALL THE SECOND ORDER PARTIAL DERIVATIVES OF $f$ EXIST AT $(x_0, y_0)$ (viz. $f_{xx}$, $f_{xy}$, $f_{yx}$, $f_{yy}$)

WE DENOTE BY $\Delta f(x_0, y_0)$, THE HESSIAN OF $f$ AT $(x_0, y_0)$:

$$\Delta f(x_0, y_0) := \begin{vmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{yx}(x_0, y_0) & f_{yy}(x_0, y_0) \end{vmatrix}$$

$$= (f_{xx} f_{yy} - f_{xy} f_{yx})(x_0, y_0). \quad \text{(2×2 DETERMINANT)}$$

SUPPOSE $f : U \to \mathbb{R}$ $(U \subseteq \mathbb{R}^2)$ AND $(x_0, y_0)$ IS AN INTERIOR POINT. SUPPOSE $f_{xx}$, $f_{xy}$, $f_{yy}$ EXIST AND ARE CONTINUOUS ON $B_\delta(x_0, y_0)$ FOR SOME $\delta > 0$. SUPPOSE FURTHER THAT

(GRAD.) $\leftarrow \nabla f(x_0, y_0) = (0, 0)$. THEN

(i) $f$ HAS A LOCAL MAXIMUM AT $(x_0, y_0)$ IF $f_{xx}(x_0, y_0) < 0$ AND $\Delta f(x_0, y_0) > 0$. (HESSIAN)

(ii) $f$ HAS A LOCAL MINIMUM AT $(x_0, y_0)$ IF $f_{xx}(x_0, y_0) > 0$ AND $\Delta f(x_0, y_0) > 0$

(iii) $f$ HAS A SADDLE POINT AT $(x_0, y_0)$ IF $\Delta f(x_0, y_0) < 0$.

IF $\Delta f(x_0, y_0) = 0$, THEN NO DEFINITE CONCLUSION CAN BE DRAWN. IN THIS CASE ONE HAS TO TRY SOMETHING ELSE.

# EXAMPLE

$f(x,y) = 4xy - x^4 - y^4$. FIND MAX/MIN

$$f_x = 4y - 4x^3 \qquad f_y = 4x - 4y^3$$

$f$ IS DIFF. EVERYWHERE. ($U = \mathbb{R}^2$)

CRITICAL POINTS: $\quad y = x^3, \; x = y^3$

$$\Rightarrow y = y^9 = y(y^8 - 1) = 0$$

$$\Rightarrow \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$f_{xx} = -12x^2, \; f_{yy} = -12y^2 \quad f_{xy} = 4 = f_{yx}$

$$\Delta f = \begin{vmatrix} -12x^2 & 4 \\ 4 & -12y^2 \end{vmatrix}$$

AT $(0,0)$: $\quad \Delta f = -16 \Rightarrow (0,0)$ IS SADDLE

$(1,1)$: $\quad \Delta f = 128 > 0; \; -12x^2\Big]_{x=1} = -12 < 0$

$\Rightarrow (1,1)$ IS LOCAL MAX.

$(-1,-1)$, $\Delta f = 128 > 0$, $(-1,-1)$ IS LOCAL MAX

# MAX/MIN WITH CONSTRAINTS

## THE METHOD OF LAGRANGE MULTIPLIERS

CONSIDER THE FOLLOWING PROBLEMS:

- FIND THE CLOSEST POINT ON THE PLANE $2x + y - 3z = 5$, TO THE ORIGIN.

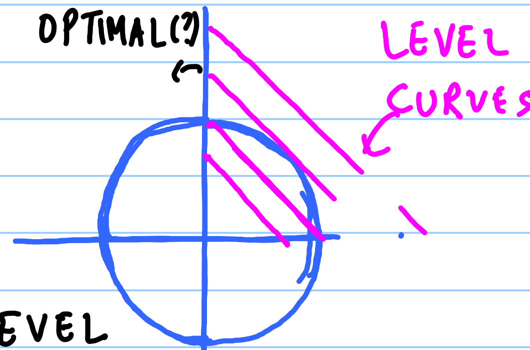
- A SATELLITE IN THE SHAPE OF THE ELLIPSOID $4x^2 + y^2 + 4z^2 = 16$ ENTERS THE EARTH'S SURFACE AND IT'S SURFACE BEGINS TO HEAT. THE TEMPERATURE ON THE SURFACE AT $(x,y,z)$ IS GIVEN BY $T(x,y,z) = 8x^2 + xyz - 16z + 600$ CELSIUS. WHICH IS THE HOTTEST POINT ON THE SATELLITE?

THESE PROBLEMS ARE INSTANCES OF CONSTRAINED MAX/MIN PROBLEMS.

FIND MAX $f(x,y,z)$, SUBJECT TO $g(x,y,z) = 0$.

EXAMPLE:

FIND MAX $X+Y$ SUBJECT TO $X^2+Y^2=1$.




NOTE THAT AT THE MAXIMAL VALUE OF $X+Y$, THE 'CONSTRAINT' AND THE LEVEL CURVE '$X+Y$' ARE TANGENTIAL.

LET $(x_0, y_0) \in \mathbb{R}^2$, AND

$f, g : B_r(x_0, y_0) \to \mathbb{R}$ SATISFYING

- $f_x, f_y, g_x, g_y$ ARE CONTINUOUS AT $(x_0, y_0)$.
- $g(x_0, y_0) = 0$, $\nabla g(x_0, y_0) \neq (0,0)$
- $f$ HAS A LOCAL EXTREMUM AT $(x_0, y_0)$ WHEN RESTRICTED TO THE LEVEL CURVE

$$\mathcal{C} = \{(x,y) \mid g(x,y) = 0\}.$$

THEN $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ FOR SOME $\lambda \in \mathbb{R}$.

FOR THE PROBLEM

MAX/MIN $f(x,y)$ SUBJECT TO

$g(x,y)=0$, CONSIDER

$$F(x,y,\lambda) = f(x,y) - \lambda g(x,y).$$

$$S_1 = \{(x_0,y_0) \mid \nabla F(x_0,y_0) = (0,0)\}$$

$$S_2 = \{(x_0,y_0) \mid g(x_0,y_0) = 0 \text{ AND } f_x(x_0,y_0) \text{ OR } f_y(x_0,y_0) \text{ DNE OR } \nabla g(x_0,y_0) = (0,0)\}.$$

CHECK ALL POINTS IN $S_1 \cup S_2$ FOR MAX/MIN VALUES.

# EXAMPLES

MIN : $x^2+y^2+z^2$ SUBJECT TO $2x+y-3z=5$.

$f(x,y,z) = x^2+y^2+z^2$; $g(x,y,z) = 2x+y-3z-5$

$F = f - \lambda g \quad (F \equiv F(x,y,z;\lambda))$

$\frac{\partial F}{\partial x} = 0 \Rightarrow 2x - 2\lambda = 0$; $\frac{\partial F}{\partial y} = 0 \Rightarrow 2y - \lambda = 0$

$\frac{\partial F}{\partial z} = 0 \Rightarrow 2z + 3\lambda = 0$ $\frac{\partial F}{\partial \lambda} = 0 \Rightarrow 2x+y-3z=5$.

$x=\lambda,\ y = \frac{\lambda}{2},\ z = \frac{-3\lambda}{2} \Rightarrow \frac{5\lambda}{2} + \frac{9\lambda}{2} = 5 \Rightarrow \lambda = \frac{5}{7}.$

MAX. $T(x,y,z) = 8x^2 + xyz - 16z + 600$ SUBJECT TO $4x^2 + y^2 + 4z^2 = 16$.

$f = 8x^2 + xyz - 16z + 600$, $g = 4x^2 + y^2 + 4z^2 - 16$

$F = f - \lambda g \quad (F(x,y,z,\lambda))$

$\nabla f = (16x + yz,\ xz,\ xy - 16)$

$\nabla g = (8x,\ 2y,\ 8z) = (0,0,0)$ IFF $x = y = z = 0$

FIND $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}, \frac{\partial F}{\partial \lambda} = 0$, AND SOLVE.

# MULTIPLE CONSTRAINTS

MAX $f(x,y,z)$ SUBJECT TO

$g_1(x,y,z)=0,\ g_2(x,y,z)=0,\ \dots,\ g_p(x,y,z)=0.$

IN THIS CASE WE CONSIDER

$$F(x,y,z;\lambda_1,\lambda_2,\dots,\lambda_p) := f-\lambda_1 g_1-\lambda_2 g_2-\dots-\lambda_p g_p$$

THEN WE PROCEED AS IN THE SINGLE CONSTRAINT CASE.

MIN. $x^2+y^2+z^2$ SUBJECT TO $x+y+z=1$ AND $3x+2y+z=6$.

$f=x^2+y^2+z^2,\quad g_1=x+y+z-1,\quad g_2=3x+2y+z-6$

$$\nabla f=\begin{pmatrix}2x\\2y\\2z\end{pmatrix}\qquad \nabla g_1=\begin{pmatrix}1\\1\\1\end{pmatrix}\qquad \nabla g_2=\begin{pmatrix}3\\2\\1\end{pmatrix}$$

$F=f-\lambda g_1-\mu g_2.$

SET $\nabla F=(0,0,0,0)$ AND SOLVE FOR $(x,y,z,\lambda,\mu)$. (EXERCISE)

# MAX/MIN ON BOUNDED REGIONS

EXAMPLE: FIND ALL TRIANGLES IN WHICH THE PRODUCT OF THE SINES OF THE ANGLES IS MAXIMUM.

IF THE ANGLES ARE $x, y, \pi-(x+y)$, WE WISH TO MAXIMIZE

$$f(x,y) = \sin x \sin y \sin(x+y) \quad \text{IN THE REGION}$$

$$R = \{(x,y) \mid 0 < x, y < \pi,\ 0 < x+y < \pi\}.$$

CONSIDER $\bar{R} = \{(x,y) \mid 0 \le x, y \le \pi,\ 0 \le x+y \le \pi\}$

$\bar{R}$ IS CLOSED (WHY? PROVE!), AND $f$ IS DIFFERENTIABLE IN $R$ AND CONTINUOUS ON $\bar{R}$ (WHY? CHECK)

$$\frac{\partial f}{\partial x} = \sin y\left[\cos x \sin(x+y) + \sin x \cos(x+y)\right]$$

$$= \sin y \ \sin(2x+y) = 0.$$

SIMILARLY $\frac{\partial f}{\partial y} = 0 \Rightarrow \sin x \sin(x+2y) = 0.$

CHECK THAT $\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right)$ IS THE <u>ONLY</u> POINT OF MAXIMUM IN $\bar{R}$. (EXERCISE)