MONOTONICITY

SAY f IS INCREASING IF $f(x) \leq f(y)$ WHENEVER $x \leq y$. IF $f(x) < f(y)$ FOR $x < y$, SAY f IS STRICTLY INCREASING. A SIMILAR DEFINITION FOR DECREASING/STRICTLY DECREASING FUNCTIONS

SUPPOSE $f : [a,b]$ IS CONTINUOUS AND DIFFERENTIABLE IN (a,b).

IF $c \in (a,b)$ AND $f'(c) \geq 0$ THEN $\exists \delta > 0$ SUCH THAT

$$f(c) \geq f(x) \quad \forall \quad x \in (c-\delta, c)$$
$$f(c) \leq f(x) \quad \forall \quad x \in (c, c+\delta)$$

CONSEQUENCES

IF $f'(x) \geq 0 \ \forall \ x \in (a,b)$ THEN f IS INCREASING. IF $f'(x) > 0 \ \forall x$, f IS STRICTLY INCREASING.

PROOF:

SUPPOSE $f'(x) = \alpha > 0$.

LET $c < d$ WE NEED TO SHOW

$$f(c) < f(d).$$

LET $S = \{ c < y \leq d \mid f(y) > f(c) \}$

BY THE PRECEDING FACT, COUPLED WITH $f'(c) > 0$

$(c, c+\delta) \subset S$ FOR SOME $\delta > 0 \Rightarrow S \neq \emptyset$

S IS BOUNDED, SO LET $\alpha = LUB(S)$

SINCE $\alpha = LUB(S)$ AND $f(y) > f(c) \ \forall \ y \in S$

$$f(\alpha) \geq f(c)$$

BUT $f'(\alpha) > 0 \Rightarrow \exists \ \delta' > 0$ S.T.

$y \in (\alpha, \alpha + \delta')$ SATISFY $f(y) \geq f(\alpha)$

$$\Rightarrow f(y) > f(c) \quad \forall \ y \in (\alpha, \alpha + \delta')$$

THIS CONTRADICTS THAT $y = LUB(S)$ $\boxed{/\!/}$

🚩 IF $f:[a,b] \to \mathbb{R}$ IS CONTINUOUS AND f IS DIFFERENTIABLE IN (a,b), AND IF $x_0 \in (a,b)$ IS AN EXTREMUM POINT, i.e., A POINT WHERE f ATTAINS MAXIMUM/MINIMUM IN $[a,b]$, THEN $f'(x_0) = 0$.

🚩 x_0 IS A POINT OF LOCAL MAXIMA/MINIMA IF THERE EXISTS $\delta > 0$ s.t x_0 IS A POINT OF MAXIMUM/MINIMUM OF f WHEN RESTRICTED TO $[x_0-\delta, x_0+\delta]$

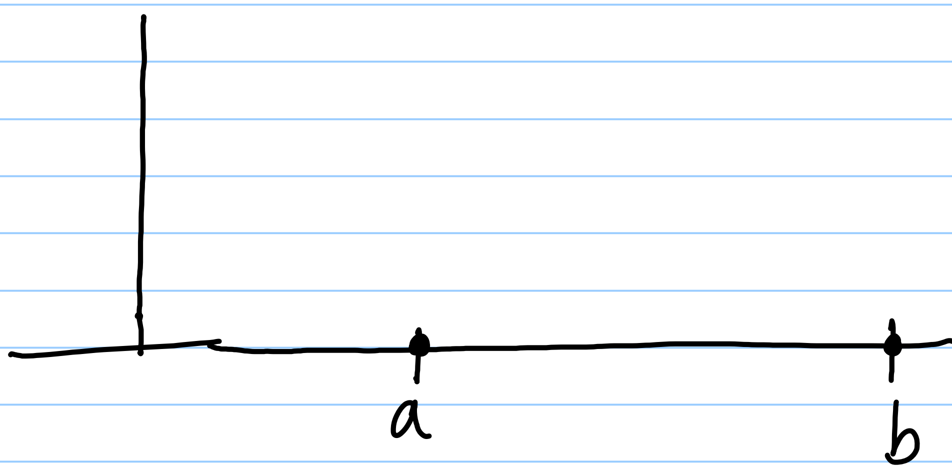
🚩 THE PREVIOUS OBSERVATION HOLDS FOR LOCAL MAXIMA/MINIMA, i.e., IF $x_0 \in (a,b)$ IS A POINT OF LOCAL MAX/MIN, AND $f'(x_0)$ EXISTS, THEN

$$f'(x_0) = 0.$$

ROLLE'S THEOREM

🚩 SUPPOSE $f:[a,b] \to \mathbb{R}$ IS CONTINUOUS AND DIFFERENTIABLE IN (a,b). IF $f(a) = f(b)$ THEN THERE EXISTS $\xi \in (a,b)$ SUCH THAT

$$f'(\xi) = 0.$$



SINCE $f(a) = f(b)$, a AND b WOULD BE MAX/MIN SIMULTANEOUSLY $\Leftrightarrow$ f = CONST

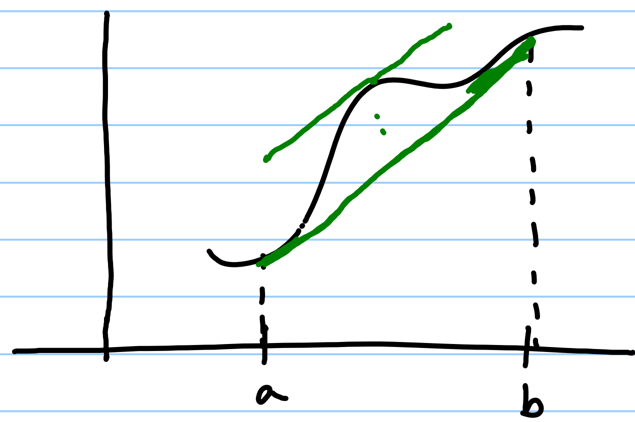
IN ANY OTHER CASE, EITHER A MAX. OR A MIN POINT MUST BE AN INTERNAL POINT IN (a,b)

$\blacksquare$

MEAN VALUE THEOREM

SUPPOSE $f:[a,b]\to\mathbb{R}$ IS CONTINUOUS AND DIFFERENTIABLE IN (a,b). THEN THERE EXISTS $\xi\in(a,b)$ SUCH THAT

$$f'(\xi) = \frac{f(b)-f(a)}{b-a}$$



🚩 IF $f, g:[a,b]\to\mathbb{R}$ ARE CONTINUOUS AND ARE DIFFERENTIABLE ON (a,b) THEN THERE EXISTS $\xi\in(a,b)$ SUCH THAT

$$\big(f(b)-f(a)\big)g'(\xi) = \big(g(b)-g(a)\big)f'(\xi)$$

(CAUCHY MVT) $\qquad$ ($g(x)=x \Rightarrow$ USUAL MVT)

CONSEQUENCES OF M.V.T.

🚩 IF $f:(a,b)\to\mathbb{R}$ AND $f'(x)=0 \ \forall x$,

THEN $f(x)=C$ (CONSTANT)

$x<y$

$$\frac{f(y)-f(x)}{y-x}=f'(\xi) \quad \text{FOR } \xi\in(x,y)$$

$$=0 \Rightarrow f(y)=f(x)$$

🚩 IF $f'(x)=g'(x) \ \forall x\in(a,b)$, THEN

$$f(x)=g(x)+C \ \forall x$$

🚩 APPROXIMATIONS:

$f(x)=\sqrt{x}$, $x\in[n,n+1]$.

$$f(n+1)-f(n)=f'(\xi) \quad \xi\in(n,n+1)$$

$$\sqrt{n+1}-\sqrt{n}=\frac{1}{2\sqrt{\xi}}$$

IF n IS 'LARGE' $\frac{1}{2\sqrt{n+1}}<\frac{1}{2\sqrt{\xi}}<\frac{1}{2\sqrt{n}}$.

$$\sqrt{n}<\sqrt{n+1}<\sqrt{n}+\frac{1}{2\sqrt{n}}$$

MAXIMA / MINIMA

🚩 A POINT c IS CALLED A CRITICAL POINT OF f IF IT SATISFIES ONE OF THE FOLLOWING: (HERE f IS CONTINUOUS)

(i) f IS <u>NOT</u> DIFFERENTIABLE AT c

OR

(ii) $f'(c) = 0$.

LOCAL EXTREMA

LET $f'(c) = 0$

SUPPOSE $f : [a,b] \to \mathbb{R}$ IS CONTINUOUS AND DIFFERENTIABLE ON (a,b). IF FOR SOME $\delta > 0$

🚩 IF $f'(x) \geq 0 \ \forall \ x \in (c-\delta, c)$ AND $f'(x) \leq 0 \ \forall \ x \in (c, c+\delta)$ THEN f HAS A LOCAL MAXIMUM AT c. (

🚩 IF $f'(x) \leq 0 \ \forall \ x \in (c-\delta, c)$ AND $f'(x) \geq 0 \ \forall \ x \in (c, c+\delta)$, f HAS A LOCAL MINIMUM AT c.

(FIRST DERIVATIVE TEST)

2^{ND} DERIVATIVE

SUPPOSE $f : (a,b) \to \mathbb{R}$ IS DIFFERENTIABLE.

CONSIDER $f' : (a,b) \to \mathbb{R}$ WHICH GIVES

THE DERIVATIVE OF f AT EACH POINT

IF THIS FUNCTION f' IS DIFFERENTIABLE

ITS DERIVATIVE WOULD BE THE

2^{ND} DERIVATIVE OF f.

SUPPOSE f HAS 2^{ND} DERIVATIVE.

IF $c \in (a,b)$ SATISFIES

- $f'(c) = 0$
- $f''(c) < 0$

THEN c IS A POINT OF LOCAL MAXIMUM

IF $c \in (a,b)$ SATISFIES

- $f'(c) = 0$
- $f''(c) > 0$

c IS A POINT OF LOCAL MINIMUM

(2^{ND} DERIVATIVE TEST)

🚩 IF $f''(c) = 0$, WE CANNOT CONCLUDE ANYTHING.