# 3 Forced vibration

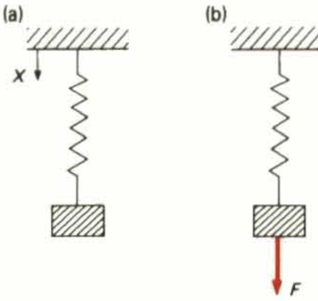


*Figure 22*

Forced vibration is the result of continuous external stimulus, in contrast to natural vibration which, once started, is left alone. It is an observed fact of engineering that assemblies of components and structures with generous safety factors against static loads will sometimes fail catastrophically when subjected to even quite mild forced vibration. To explain the reason for this I shall turn to our undamped model again, and try to analyse the effect that forced vibration will have. You should recall that most structures have quite low damping, so the undamped model is justifiable as a first approach. Figure 22 shows two ways by which forced vibration can be applied to the system – either by an externally applied force $F$ or by the motion $X$ of a previously steady mounting. In practice these vibration sources may be very complex but it is sufficient here to consider sinusoidal ones. This kind of motion is often termed *forced harmonic vibration*.

**forced harmonic vibration**

## 3.1 Forced harmonic vibration by ground motion

**ground motion**

Consider the moving fixture first, which might represent say a building suffering an earthquake, or a carburettor mounted on a vibrating engine, or an instrument panel mounted on a vibrating aircraft body. The fixture motion, or *ground motion*, with amplitude $X_0$ and frequency $\Omega$ rad s$^{-1}$, has displacement

$$X = X_0 \sin \Omega t$$

Ground motion is sometimes referred to as displacement excitation. The 'mass' displacement is $x$, so the corresponding spring extension from the equilibrium position at any particular instant (right-hand side of Figure 23), is $x - X$, and the equation of motion for the mass $m$, along $x$, is:

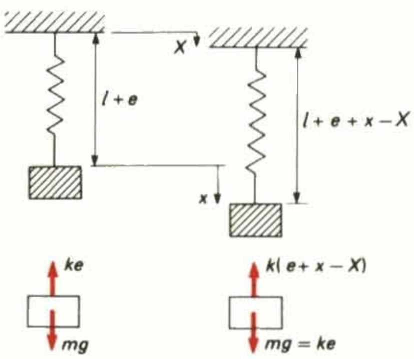


*Figure 23*

$$R_x = -k(x - X) = ma = m\ddot{x}$$

so $$m\ddot{x} + kx = kX$$

and $$m\ddot{x} + kx = kX_0 \sin \Omega t$$

The solution to this equation will be the motion of the model, which will be our estimate of corresponding real motions. The complete solution to this equation is a combination of natural vibration and forced vibration, and this is discussed further in Section 5.1. For the present I will confine my attention to the forced vibration. In mathematical terms the solution we are looking for is the Particular Integral of the equation of motion. Experience suggests that the response has the same frequency as that of the ground motion, so we will try as a possible solution

$$x = A \sin \Omega t$$

for which $\ddot{x} = -\Omega^2 A \sin \Omega t$.

Substituting into the differential equation gives

$$m(-\Omega^2 A \sin \Omega t) + k(A \sin \Omega t) = kX_0 \sin \Omega t$$

so $$(-m\Omega^2 + k) A \sin \Omega t = kX_0 \sin \Omega t$$

If the mounting is vibrating, then in general $\sin \Omega t$ is non-zero, so that I can cancel it from both sides of the equation giving

$$(-m\Omega^2 + k) A = kX_0$$

$$A = \frac{kX_0}{k - m\Omega^2} = \frac{X_0}{1 - \dfrac{m}{k}\Omega^2}$$

We have seen therefore that the trial solution is valid if $A$ has this value. This is what we wanted to know – the amplitude $A$ of the response. I can simplify the result by remembering that the undamped natural frequency $\omega = \sqrt{(k/m)}$, that is $\omega^2 = k/m$, so that

$$\frac{A}{X_0} = \frac{1}{1 - (\Omega/\omega)^2}$$

This tells us how the response amplitude $A$ compares with the mounting amplitude $X_0$. It depends upon how the driving frequency $\Omega$ compares with the natural frequency $\omega$. Expressing the equation in the form of ratios $A/X_0$ and $\Omega/\omega$ enables us to examine the important features of this response without knowing details of the input amplitude or frequency, other than in comparison with the natural frequency. This equation can be applied to mechanical vibrations ranging from quartz crystals at many thousands of hertz to ships at only a fraction of one hertz. It can even be applied to electrical oscillations at millions of hertz. What we need to know now is the shape of the graph showing how $A/X_0$ depends on $\Omega/\omega$.

***SAQ 10***

What are the units of (a) $\Omega/\omega$, and (b) $A/X_0$? (c) Check the units of the equation for $A/X_0$.

***SAQ 11***

(a) Evaluate $A/X_0$ for the following values of $\Omega/\omega$. Note that the amplitude is always positive by definition, so you can simply ignore a minus sign in front of the answers.

0, 0.4, 0.8, 0.9, 1.0, 1.1, 1.2, 1.4, 1.8, 2.0, 3, 10, 100

(b) Sketch the graph of $A/X_0$ for values of $\Omega/\omega$ between 0 and 3.

(c) What value of $\Omega/\omega$ gives the largest value of $A/X_0$?

(d) What do you think would be the result of applying a vibration of frequency $\Omega = \omega$ to a mechanical assembly?

The ratio $\Omega/\omega$ is called the *frequency ratio*. The ratio $A/X_0$ is called the *transmissibility*, $T$, because it is the fraction of the input amplitude transmitted to the other side of the spring,

**frequency ratio**
**transmissibility**

$$A = TX_0$$

As was implied in SAQ 11 part (a), the amplitude $A$ and hence the transmissibility $T$ are always specified as positive and this is a matter to which I will return later in Section 4.1 in a discussion of phase.

If $\Omega$ is very low indeed then we would naturally expect the whole spring and mass to move up and down slowly without any significant change of spring length – the acceleration of the mass is so small that the force required to provide it is negligible, so the extra spring extensions are negligible. The transmissibility would be 1.0. At very high input frequencies, the force applied to the mass will hardly move it because in a very short time it reverses its direction. The transmissibility would be zero. The remarkable feature of the theoretical model is that in between, at frequencies close to the natural frequency, there is a very large transmissibility. At exactly $\Omega = \omega$ it is infinity. This large response is called *resonance* (Figure 24) and $\Omega = \omega$ is the *resonance frequency*.

**resonance**
**resonance frequency**

Of course, we would not expect a corresponding infinite response from a real system – just a very large response quite possibly leading to breakage. Even for the model an infinite amplitude would not be expected instantly – the amplitude takes time to build up. One reason for this discrepancy is that we have been analysing the steady-state response only. Even when there is forced vibration there can be a natural vibration occurring at the same time. This is triggered by the initial conditions of the forced vibration. In practice it usually dies away because of damping, to leave

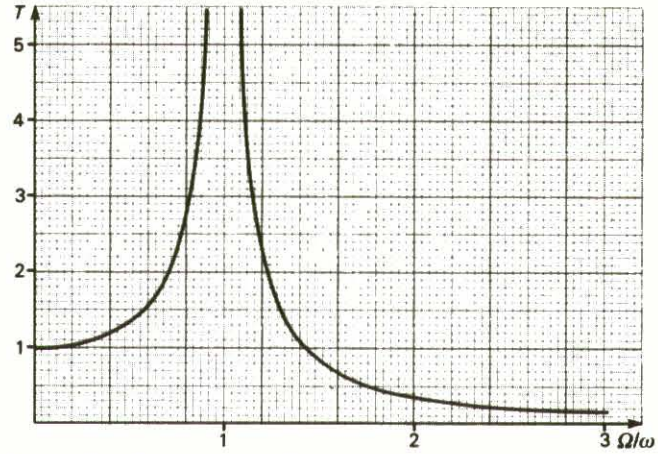

*Figure 24 Transmissibility (no damping)*

**transient vibration**

only the steady-state forced response. The natural vibration is in this context called a *transient vibration*. The transients are discussed a little further later in this Unit.

There is another reason why the infinite response of the model does not occur in practice. There is damping in any real system. Close to resonance the large amplitude of response means high speeds, and so the damping can no longer be neglected. Close to resonance the response depends on the damping, so I shall look at this in Section 4. Away from resonance light damping has little effect. Since the mechanical engineer is usually mainly interested in keeping well away from resonance, the precise details of the response at resonance are often of only secondary interest. In other words, we are usually more interested in making sure that a system does not resonate, than in analysing exactly how it will break if we allow it to resonate.

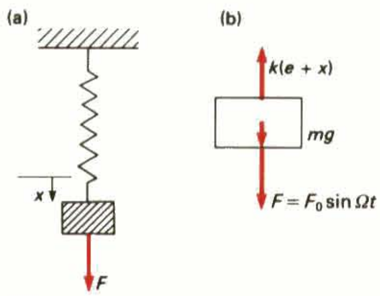

*Figure 25*

## 3.2 Forced harmonic vibration by applied force

What about the other way in which the forced harmonic vibration can be caused – namely by the application of a harmonic force directly to the mass element (Figure 25a)? The free-body diagram is shown in Figure 25(b), where

$$F = F_0 \sin \Omega t$$

**force amplitude**

$F_0$ is called the *force amplitude* – it is the 'amplitude' of the applied force, the peak value of the sinusoidal variation (measured in newtons of course). The ground is now considered very rigid so that its motion can be neglected.

The equation of motion along $x$ is:

$$R_x = F_0 \sin \Omega t - kx = m\ddot{x}$$

so that

$$m\ddot{x} + kx = F_0 \sin \Omega t$$

When the vibration was applied by shaking the support the equation was:

$$m\ddot{x} + kx = kX_0 \sin \Omega t$$

The only difference here is that $kX_0$ has been replaced by $F_0$. The analysis will, therefore, be equivalent, and the old result

$$\frac{A}{X_0} = \frac{1}{1 - (\Omega/\omega)^2}$$

becomes instead

$$\frac{A}{(F_0/k)} = \frac{1}{1-(\Omega/\omega)^2}$$

Note that $F_0/k$ would be the extension of the spring resulting from the smooth application of a steady (non-oscillating) force of magnitude $F_0$. The expression $A/(F_0/k)$ is called the *magnification ratio*, $M$, because it is the ratio of the response amplitude to $F_0/k$:

**magnification ratio**

$$A = M(F_0/k)$$

The shape of the magnification ratio and the transmissibility graphs are the same, but only for this special case of negligible damping. The two are different in principle. The magnification ratio relates the motion of an object to the force acting on it. The transmissibility relates the input on one side of the spring to the resulting motion on the other side. Once again as $A$ is always positive the magnification ratio $M$ is also always taken to be positive. The significance of the minus sign in the above definition for $M$, when $\Omega/\omega > 1$, will become apparent in Section 4.1 in the context of phase.

Notice that in this forced vibration produced by an applied force $F_0 \sin \Omega t$ a force $P$ is transmitted through the spring to the fixed support. This force is

$$P = kx = kA \sin \Omega t = P_0 \sin \Omega t$$

where its amplitude is

$$P_0 = kA = \frac{F_0}{1-(\Omega/\omega)^2} = TF_0$$

and $$T = \frac{1}{1-(\Omega/\omega)^2}$$

is the transmissibility, which we met first in Section 3.1.

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***SAQ 12***

An 800 g machine component mounted on springs is known to have a natural frequency of 25 rad $s^{-1}$ and little damping (from observation of its natural vibration). Its base is expected to be vibrated at a frequency of 30 rad $s^{-1}$ with amplitude 2 mm. Estimate (a) the spring stiffness, (b) the frequency ratio, (c) the transmissibility, (d) the response amplitude.

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***SAQ 13***

A delicate instrument of mass 400 g is mounted on springs to isolate it from the severe vibration of the framework which must support it. The expected framework vibration is 2 mm amplitude at 30 Hz. The maximum acceptable instrument amplitude is 0.4 mm. Estimate (a) the maximum acceptable transmissibility, (b) (from the graph in Figure 24) the acceptable range of natural frequency of the instrument on its mounting, (c) the permissible mounting spring stiffness.

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***SAQ 14***

A 140 kg engine is subject to an approximately sinusoidal force of amplitude 800 N and frequency 50 Hz (3000 rev/min). The mountings are rubber blocks with stiffness 60 kN $m^{-1}$ with little damping ($\zeta < 0.02$). (a) Estimate the amplitude of vibration of the engine. (b) Estimate the amplitude of the force applied to the car through the mountings.

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**Summary of Section 3**

If a mechanical assembly capable of natural vibration is stimulated by an external source of vibration then it will vibrate. The frequency of the response is the same as that of the forcing vibration. The amplitude of the response depends on the forcing amplitude and the frequency ratio $\Omega/\omega$.

When the source of vibration is a force acting on the object, the response amplitude is

$$A = MF_0/k$$

If the source is displacement of the ground with amplitude $X_0$, the response amplitude is

$$A = TX_0$$

When $\Omega = \omega$ the assembly is in resonance and the response amplitude becomes very large (infinite in terms of the undamped model).