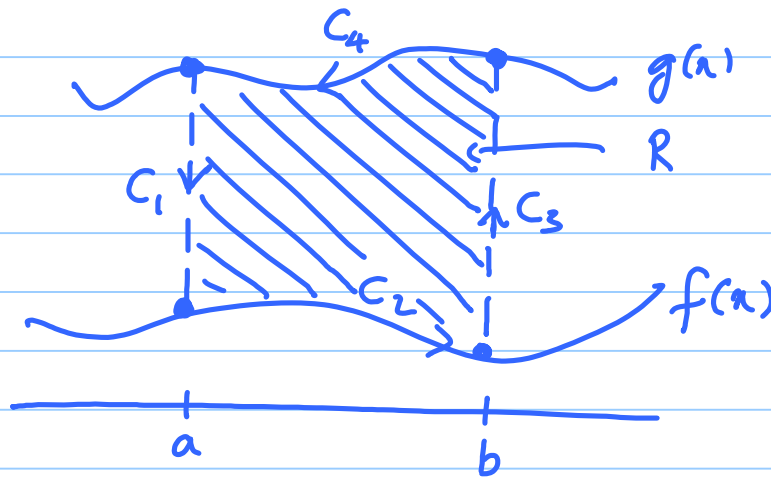


GREEN'S THEOREM

SUPPOSE $R \subseteq \mathbb{R}^2$ IS A TYPE I ELEMENTARY

REGION: (VERTICALLY SIMPLE REGION)

$$R = \{ (x, y) \mid a \leq x \leq b, f(x) \leq y \leq g(x) \}.$$



SUPPOSE C IS THE BOUNDARY OF R . IT IS

DENOTED BY WRITING $\partial R = C$.

AS INDICATED ABOVE, WRITE

$$C = C_1 \cup C_2 \cup C_3 \cup C_4, \text{ WHERE } C_i \text{ ARE}$$

ORIENTED SO THAT C IS COUNTER CLOCKWISE.

A REGION $R \subseteq \mathbb{R}^2$ IS CALLED A SIMPLE REGION

IF IT IS A TYPE I AS WELL AS TYPE II

ELEMENTARY REGION.



LET $M, N: U \rightarrow \mathbb{R}$ BE CONTINUOUSLY $(F = M\vec{i} + N\vec{j})$

DIFFERENTIABLE SCALAR FIELDS. HERE $U \subseteq \mathbb{R}^2$.

SUPPOSE R IS A SIMPLE REGION WITH A

SMOOTH BOUNDARY C . SUPPOSE $R, C \subseteq U$. THEN

$$\int_C F \cdot dr = \oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

EXAMPLE

$$F(x, y) = M\vec{i} + N\vec{j}, \quad M := -x^2y, \quad N := xy^2.$$

$$R = \{(x, y) \mid x^2 + y^2 \leq a^2\}$$

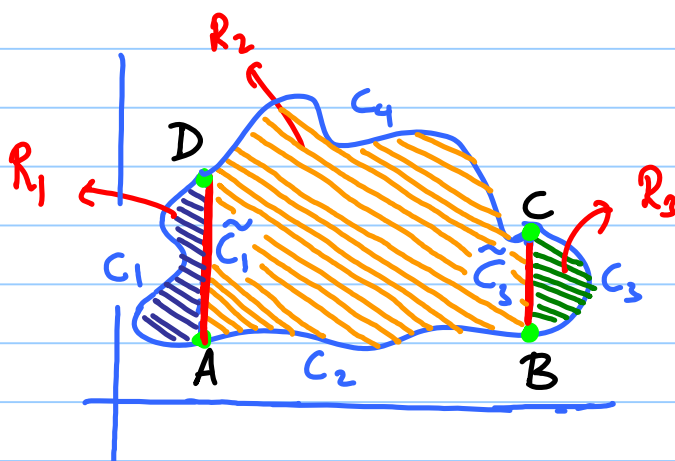
$$\text{LET } r(t) = a \cos t \vec{i} + a \sin t \vec{j}, \quad 0 \leq t \leq 2\pi.$$

$$\begin{aligned} \text{THEN } \oint_C M dx &= \int_0^{2\pi} -(a \cos t)^2 (a \sin t) (-a \sin t) dt \\ \oint_C N dy &= \int_0^{2\pi} (a \cos t) (a \sin t)^2 (a \cos t) dt \end{aligned}$$

$$\text{AND } \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \iint_R x^2 + y^2 dx dy.$$

CHECK THAT THESE ARE EQUAL

BEYOND SIMPLE REGIONS



REGION R_1 ; REGION R_2 ; REGION R_3 .

LET US APPLY GREEN'S THEOREM ON

R_1, R_2, R_3 :

$$\iint_{R_i} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \oint_{C_i} M dx + N dy$$

$i = 1, 2, 3$

SINCE THE INTEGRALS OVER THE

'VERTICAL LINES' CANCEL EACH OTHER,

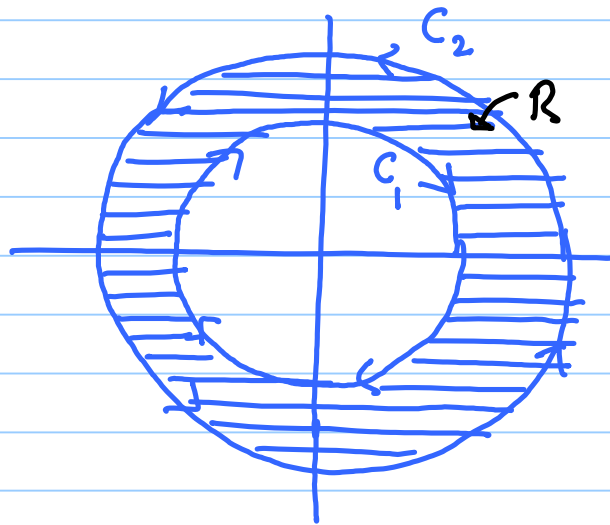
$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \oint_C M dx + N dy$$

GREEN'S THEOREM HOLDS FOR 'ALL' REGIONS
BOUNDED BY SIMPLE CLOSED CURVES,

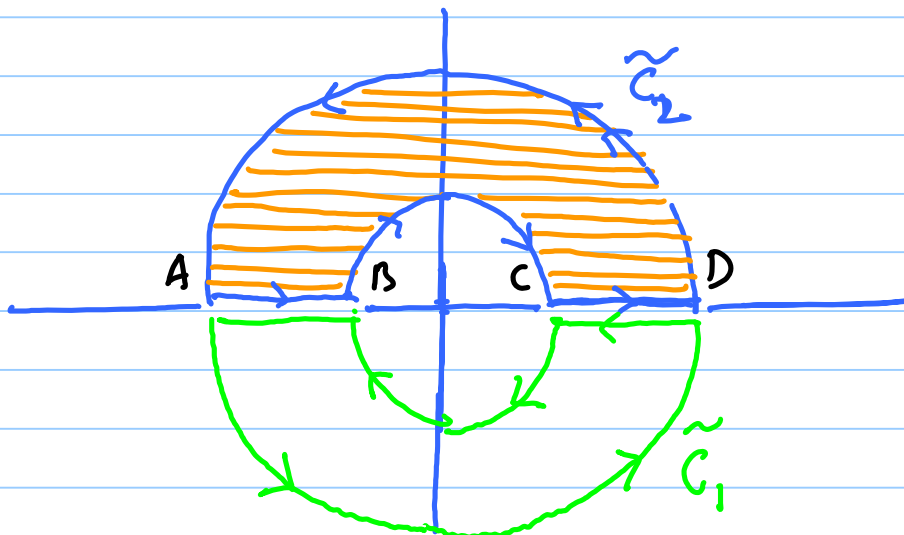
THIS EXTENDS EVEN FURTHER

CONSIDER

$$R = \{ (x,y) \mid a^2 < x^2 + y^2 < b^2 \}$$



THIS MAY BE RENDERED AS FOLLOWS:



So, we again have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \oint_{C_1} M dx + N dy + \oint_{C_2} M dx + N dy$$

GREEN'S THEOREM CAN BE EXTENDED SIMILARLY TO DOMAINS WITH A FINITE NUMBER OF 'HOLES'.

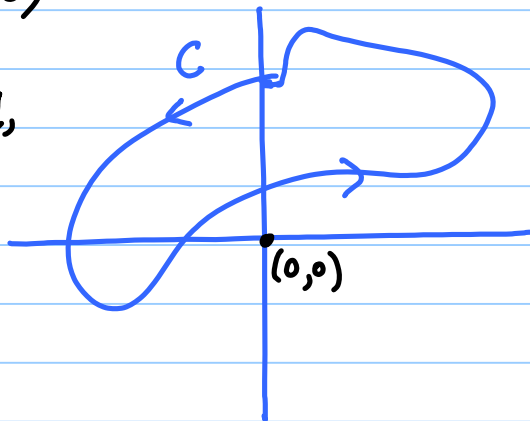
EXAMPLE

$$F(x, y) = \frac{-y}{x^2 + y^2} \vec{i} + \frac{x}{x^2 + y^2} \vec{j} \quad (x, y) \neq (0, 0)$$

LET C BE A PIECEWISE SMOOTH CURVE

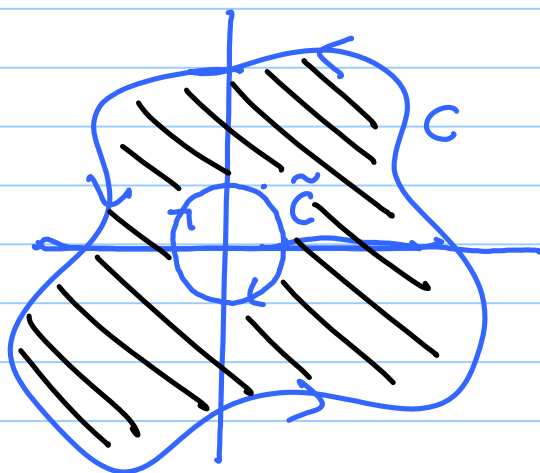
NOT CONTAINING $(0, 0)$

BY GREEN'S THEOREM,



$$\oint_C \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = 0$$

ON THE OTHER HAND, IF C CONTAINS $(0,0)$:



LET \tilde{C} BE A CIRCLE $x^2 + y^2 = r^2$ FOR SOME

SUITABLE $r > 0$ SO THAT \tilde{C} IS CONTAINED

'INSIDE' C . ORIENT \tilde{C} CLOCKWISE.

$$\text{GREEN} \Rightarrow \int_C + \int_{\tilde{C}} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = 0$$

FLUX/WORK FORMS OF GREEN

$$C: \vec{r}(s) = x(s)\vec{i} + y(s)\vec{j} \quad \text{for } s \in [0,1].$$

SUPPOSE R IS THE REGION BOUNDED BY C .

SUPPOSE THIS IS THE ARC-LENGTH PARAMETRIZATION
FOR C . (UNIT SPEED PARAMETRIZATION FOR C)

$$\vec{T}(s) = \text{UNIT TANGENT VECTOR TO } C$$

$$\text{AT } s = (x'(s), y'(s))$$

$$\begin{aligned} \vec{n}(s) &= \text{UNIT NORMAL POINTING 'OUTWARD'} \\ &= (y', -x') \end{aligned}$$

SUPPOSE $F = M\vec{i} + N\vec{j}$ IS CONTINUOUSLY

DIFFERENTIABLE, IN D AND $R, C \subseteq D$. WRITE

$$G = N\vec{i} - M\vec{j} \Rightarrow \text{div}(G) =$$

$$\text{AND } G \cdot \vec{n} =$$

HENCE



$$\iint_R \text{div}(G) \, dx \, dy = \oint_C (G \cdot \vec{n}) \, ds$$

(FLUX FORM OF
GREEN'S THEOREM)

SUPPOSE $F = M\vec{i} + N\vec{j}$ AS BEFORE;

$$\text{curl}(F) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \vec{k}$$


ALSO, $\oint_C M dx + N dy$

$$= \oint_C (M\vec{i} + N\vec{j}) \cdot (x'(s)\vec{i} + y'(s)\vec{j}) ds$$

=

HENCE GREEN'S THEOREM CAN BE

WRITTEN AS


$$\oint_C (F \cdot T) ds = \iint_R [\text{curl}(F) \cdot \vec{k}] dx dy$$

\parallel
 $\oint_C F \cdot dr$

THE FORMER IS THE FLUX FORM OF GREEN'S
THEOREM AND THE VERSION ABOVE IS THE
WORK FORM OF GREEN'S THEOREM.

APPLICATIONS OF GREEN

EVALUATING LINE INTEGRALS

COMPUTE $\oint_C (5 - \underbrace{xy}_{M} - y^2) dx - (2xy - \underbrace{x^2}_{N}) dy$

WHERE $C = \partial R$, $R = \{(x, y) \mid 0 \leq x, y \leq 1\}$

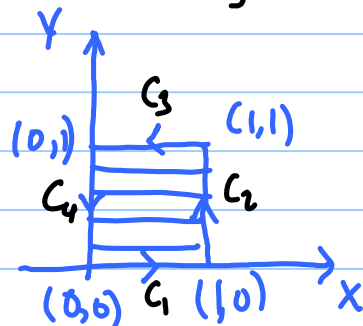
$$C = C_1 \cup C_2 \cup C_3 \cup C_4$$

$$C_1: \mathbf{r}(t) = (x(t), y(t))$$

$$\mathbf{r}(x) = (x, 0), \quad x \in [0, 1]$$

$$C_2: \mathbf{r}(y) = (1, y) \quad y \in [0, 1]$$

SIMILARLY, C_3, C_4 .



BUT GREEN $\Rightarrow \oint_C M dx + N dy = \iint_R (2x - 2y) - (-x - 2y) dA$

$$M = 5 - xy - y^2$$

$$N = x^2 - 2xy$$

$$= \int_0^1 \int_0^1 3x \, dx \, dy = \int_0^1 3x \, dx$$

$$= \frac{3}{2}$$

AREA ENCLOSED BY A CURVE

SUPPOSE C IS A SIMPLE CLOSED CURVE IN \mathbb{R}^2
ENCLOSING A REGION R . WE WISH TO
CALCULATE $\text{AREA}(R)$.

CONSIDER $M(x, y) = y$, $N(x, y) = 0$

$$\oint_C M dx + N dy \stackrel{\text{GREEN}}{=} - \iint_R dx dy$$

SIMILARLY, SUPPOSE $M(x, y) = 0$, $N(x, y) = x$

$$\oint_C M dx + N dy = \iint_R 1 dx dy$$

$$\text{HENCE, } \text{AREA}(R) = \frac{1}{2} \oint_C -y dx + x dy$$

IF C HAS PARAMETRIZATION $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$
 $a \leq t \leq b$

THEN THE LINE INTEGRAL ABOVE IS :

$$\frac{1}{2} \int_a^b [-y(t)x'(t) + x(t)y'(t)] dt$$

IF C HAS POLAR COORDINATES $x(\theta) = r(\theta) \cos \theta$

$$y(\theta) = r(\theta) \sin \theta$$

$$\theta_0 \leq \theta \leq \theta_1$$

THEN THE AREA ENCLOSED BY C EQUALS

$$\frac{1}{2} \int_C -y dx + x dy = \frac{1}{2} \int_{\theta_0}^{\theta_1} r^2 d\theta \quad (\text{CHECK?!})$$

AFTER A SMALL CALCULATION

SUFFICIENT CONDITION FOR A VECTOR FIELD

TO BE CONSERVATIVE

SUPPOSE $D \subseteq \mathbb{R}^2$ S.T. FOR ANY SIMPLE CLOSED CURVE C IN D , THE REGION R ENCLOSED BY C

SATISFIES $R \subseteq D$. SUPPOSE $F = M\vec{i} + N\vec{j}$

AND $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. DOES THERE EXIST

$\phi: D \rightarrow \mathbb{R}$ S.T. $\nabla\phi = F$?

USE GREEN'S THEOREM

SUCH A DOMAIN IS CALLED SIMPLY CONNECTED.

FORMULAS FOR ∇^2

RECALL THE LAPLACIAN OPERATOR,

$$\nabla^2(f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

WHERE f IS A SMOOTH SCALAR FIELD.

$$\text{DEFINE } \vec{F} := -\frac{\partial f}{\partial y} \vec{i} + \frac{\partial f}{\partial x} \vec{j}.$$

LET C DENOTE A SIMPLE CLOSED CURVE AND

R A REGION S.T. $C = \partial R$.

GREEN \Rightarrow

$$\iint_R (\nabla^2 f) \, dx \, dy = \iint_R \left[\overset{N}{\frac{\partial}{\partial x}} \left(\overset{M}{\frac{\partial f}{\partial x}} \right) - \frac{\partial}{\partial y} \left(-\frac{\partial f}{\partial y} \right) \right]$$

$$= \oint_C -\frac{\partial f}{\partial y} \, dx + \frac{\partial f}{\partial x} \, dy$$

$$= \oint_C \left(\frac{\partial f}{\partial y} \cdot \frac{dx}{ds} + \frac{\partial f}{\partial x} \frac{dy}{ds} \right) ds$$

$$= \oint_C \left(\frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} \right) \cdot \left(\frac{dy}{ds} \vec{i} - \frac{dx}{ds} \vec{j} \right) ds$$

$$= \oint_C (\nabla f) \cdot \vec{n} \, ds$$

WRITE $\frac{\partial f}{\partial n} := (\nabla f) \cdot \mathbf{n}$, SO THAT WE HAVE

$$\iint_R (\nabla^2 f) \, dx \, dy = \oint_C \frac{\partial f}{\partial n} \, ds$$

HENCE $\frac{\partial f}{\partial n} = 0 \Rightarrow \iint_R \nabla^2 f \, dx \, dy = 0$. GREEN'S IDENTITY



SUPPOSE ϕ IS A SCALAR FIELD AS ABOVE.

$$\phi \frac{\partial \phi}{\partial n} = \phi (\nabla \phi \cdot \mathbf{n}) = \frac{1}{2} (\nabla \phi^2) \cdot \mathbf{n}.$$

GREEN $\Rightarrow \oint_C \left(\phi \frac{\partial \phi}{\partial n} \right) ds = \oint_C \frac{1}{2} (\nabla \phi^2) \cdot \mathbf{n} \, ds$

$$(\text{FLUX})_{(\text{FORM})} = \frac{1}{2} \iint_R \text{div}(\nabla \phi^2) \, dx \, dy$$

$$\oint_C \phi \frac{\partial \phi}{\partial n} = \iint_R \text{div}(\phi \nabla \phi) \, dx \, dy = \iint_R \left[(\nabla \phi)^2 + \phi \nabla^2 \phi \right] \, dx \, dy$$

IN PARTICULAR, IF $\frac{\partial \phi}{\partial n} = 0$ AND $\nabla^2 \phi = 0$ THEN

$$\iint_R |\nabla \phi|^2 \, dx \, dy = 0 \Rightarrow \nabla \phi \cdot \nabla \phi = 0$$

$$\Rightarrow \frac{\partial \phi}{\partial x} = 0, \frac{\partial \phi}{\partial y} = 0.$$

A FUNCTION ϕ THAT SATISFIES

IS CALLED A HARMONIC FUNCTIONS